ABELIAN YANG-MILLS THEORY ON REAL TORI AND THETA DIVISORS OF KLEIN SURFACES

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CONTENTS

0. INTRODUCTION

The purpose of this paper is to determine natural theta line bundles of Klein surfaces as elements in the Grothendieck cohomology group which classifies Real line bundles in the sense of Atiyah [2].

Recall that a Klein surface is a pair (C, ι) consisting of a closed Riemann surface C and an anti-holomorphic involution $\iota : C \to C$. The topological type of a Klein surface is determined by the triple (g, r, a), where g is the genus of C, r the number of connected components of the fixed point locus C^{ι}, and a is the orientation obstruction of the ι-quotient, i.e. $a(C, \iota) = 0$ when $C/\langle \iota \rangle$ is orientable and $a(C, \iota) = 1$ when not. The Real structure ι induces a Real structure $\hat{\iota} : \mathrm{Pic}(C) \to \mathrm{Pic}(C)$ on the Picard group, given by $\hat{\iota}[\mathcal{L}] := [\iota^* \bar{\mathcal{L}}]$. The geometric theta divisor

$$\Theta := \{[\mathcal{L}] \in \mathrm{Pic}^{g-1}(C) | \ h^0(\mathcal{L}) > 0\}$$

is $\hat{\iota}$-invariant and therefore defines a natural Real holomorphic line bundle $\mathcal{L} := \mathcal{O}_{\mathrm{Pic}^{g-1}(C)}(\Theta)$ on $\mathrm{Pic}^{g-1}(C)$. There are two important families of Real holomorphic line bundles which one gets by translating $\mathcal{L}$ to $\mathrm{Pic}^0(C)$: One can either choose

a Real theta characteristic $\kappa \in \mathrm{Pic}^{g-1}(C)$ and put $\mathcal{L}_\kappa := \mathcal{O}_{\mathrm{Pic}^0(C)}(\Theta - \kappa)$, or when $C^\iota \neq \emptyset$, one can choose a point $p_0 \in C^\iota$ and define $\mathcal{L}_{p_0} := \mathcal{O}_{\mathrm{Pic}^0(C)}(\Theta - [\mathcal{O}_C((g-1)p_0)])$. Both families of theta line bundles appear naturally as determinant index bundles of certain families of perturbed Dirac operators (see section 1). The underlying differential Real line bundles $(L_\kappa, \tilde{\iota}_{\mathcal{L}_\kappa})$ and $(L_{p_0}, \tilde{\iota}_{\mathcal{L}_{p_0}})$ define elements in the Grothendieck cohomology group [10] $H^1_{\mathbb{Z}_2}(\mathrm{Pic}^0(C), \underline{S}^1(1)) \simeq H^2_{\mathbb{Z}_2}(\mathrm{Pic}^0(C), \underline{\mathbb{Z}}(1))$ of Real line bundles on the Real torus $(\mathrm{Pic}^0(C), \hat{\iota})$. In order to determine and understand these elements, we first need an explicit description of this Grothendieck cohomology group, so we need a complete classification of Real line bundles on a Real torus. This problem turned out to be more complex than we thought, and is solved in section 3.3.

The cohomology group classifying $\hat{\iota}$-Real bundles on $\mathrm{Pic}^0(C)$ comes with two natural morphisms:

$$c : H^1_{\mathbb{Z}_2}(\mathrm{Pic}^0(C), \underline{S}^1(1)) \to H^2(\mathrm{Pic}^0(C), \mathbb{Z}) \ ,$$

$$w : H^1_{\mathbb{Z}_2}(\mathrm{Pic}^0(C), \underline{S}^1(1)) \to H^1(\mathrm{Pic}^0(C)^{\hat{\iota}}, \mathbb{Z}_2) \ ,$$

defined by $c([L, \tilde{\iota}]) := c_1(L)$, $w([L, \tilde{\iota}]) := w_1(L^{\tilde{\iota}})$. It is very important to determine the first Stiefel-Whitney classes $w([L_\kappa, \tilde{\iota}_{\mathcal{L}_\kappa}])$, $w([L_{p_0}, \tilde{\iota}_{\mathcal{L}_{p_0}}])$ since these classes control to a large extend the orientability of

(1) the components of the spaces $S^d(C)^\iota$ of ι-invariant points in a symmetric power $S^d(C)$ (see section 1).
(2) certain moduli spaces in Real gauge theory. We will consider this class of applications in a forthcoming article.

Whereas the first Chern class of the theta line bundles $\mathcal{L}_\kappa$ and $\mathcal{L}_{p_0}$ is well known, and can be calculated by the Atiyah-Singer index theorems for families, there is no analogous index theorem which would compute the first Stiefel-Whitney class of the corresponding fixed point bundles. We are indebted to M. Atiyah for pointing out to us this difficulty and suggesting to study the problem in special cases.

Our strategy for computing $w([L_\kappa, \tilde{\iota}_{\mathcal{L}_\kappa}])$ and $w([L_{p_0}, \tilde{\iota}_{\mathcal{L}_{p_0}}])$ is to first determine the Appell-Humbert data of $\mathcal{L}_\kappa$, then extract $w_1(\mathcal{L}_\kappa^{\tilde{\iota}_\kappa})$ from these data and, in a third step, compare $(L_\kappa, \tilde{\iota}_{\mathcal{L}_\kappa})$ with $(L_{p_0}, \tilde{\iota}_{\mathcal{L}_{p_0}})$. The final result is a completely explicit formula for $w_1(\mathcal{L}_\kappa^{\tilde{\iota}_{\mathcal{L}_\kappa}})$ in terms of $w_1(\kappa^{\tilde{\iota}_\kappa})$, and for $w_1(\mathcal{L}_{p_0}^{\tilde{\iota}_{\mathcal{L}_{p_0}}})$ in terms of the component of C^ι in which p_0 lies.

Let us now briefly describe the content of the five sections of the article. In Section 1 we construct, using gauge theoretical techniques, two families of Dolbeault operators on a Riemann surface C, and we show that for a Klein surface (C, ι) with $C^\iota \neq \emptyset$ the corresponding determinant line bundles have natural $\hat{\iota}$-Real structures. The obtained $\hat{\iota}$-Real bundles can be identified with the underlying differentiable line bundles of $\mathcal{O}_{\mathrm{Pic}^0(C)}(\Theta - [\mathcal{O}_C((g-1)p_0)])$ and $\mathcal{O}_{\mathrm{Pic}^0(C)}(\Theta - \kappa)$. We also explain how the orientability of different components of the ι-invariant part $S^d(C)^\iota$ of the d-th symmetric power of C is controlled by the first Stiefel-Whitney class of the corresponding real line bundles on $\mathrm{Pic}^0(C)^{\hat{\iota}}$.

In section 2 we apply Grothendieck's formalism [10] of equivariant sheaf cohomology to identify the set of isomorphism classes of ι-Real line bundles on a space with involution (X, ι) with the cohomology group $H^1_{\mathbb{Z}_2}(X, \underline{S}^1(1))$. We obtain a fundamental short exact sequence for this cohomology group, which will allow us to

compute it explicitly in several important cases. The section continues with an interesting result which allows one to identify the isomorphism classes of ι-Real line bundles on a compact manifold M with involution, with the connected components of the fixed point locus of the induced involution on the moduli spaces of Yang-Mills connections on M. This result has an important complex geometric version in which the Yang-Mills moduli space is replaced by the Picard group.

The third section is dedicated to the classification of Real line bundles on a torus with involution. In this section we make use of an important remark: the classical way to construct line bundles on a torus $T = V/\Lambda$ using S^1-valued factors of automorphy associated with u-characters $(a_\lambda)_{\lambda \in \Lambda}$ is related to Yang-Mills theory. More precisely, the line bundle associated with a u-character $(a_\lambda)_{\lambda \in \Lambda}$ comes with a canonical Hermitian connection, which is Yang-Mills and whose holonomy along the loop associated a lattice element $\lambda \in \Lambda$ coincides with $\bar{a}_\lambda$. This gives an important differential geometric interpretation of the coefficients a_λ intervening in the expression of a factor of automorphy. This is relevant for our purposes, because the Stiefel-Whitney class of a real line bundle can be identified with the holonomy representation of an O(1)-connection on it. The main result of the section is a classification theorem for Real line bundles $(L, \tilde{\iota})$ on a torus with involution (T, ι), in terms of characteristic classes. A first step in the proof is an important universal *difference formula* which prescribes the jump of the Stiefel-Whitney class of the real line bundle $L^{\tilde{\iota}}\big|_{T_{[\mu]}}$ when one passes from one connected component $T_{[\mu]} \in \pi_0(T^\iota)$ to another.

Section 4 concerns Real theta line bundles on complex tori. We first use the Kobayashi-Hitchin correspondence to obtain a differential geometric interpretation of the coefficients intervening in the canonical factor of automorphy defining holomorphic line bundles on a complex torus. We continue by applying our formalism to the Riemann theta line bundle on a principally polarized abelian variety (X_Z, H_Z) associated with a point Z in the Siegel upper half space. We show that for a Real principally polarized abelian variety, either its Riemann theta line bundle, or a translate of it, is naturally a Real line bundle, and we determine explicitly the corresponding element in the Grothendieck cohomology group $H^1_{\mathbb{Z}_2}(X_Z, \underline{S}^1(1))$. The section continues with a subsection dedicated to theta line bundles on Klein surfaces which contains our main results: we determine explicitly the elements in the Grothendieck group $H^1_{\mathbb{Z}_2}(\mathrm{Pic}^0(C), \underline{S}^1(1))$ corresponding to the $\hat{\iota}$-Real line bundles $\mathcal{O}_{\mathrm{Pic}^0(C)}(\Theta - \kappa)$, $\mathcal{O}_{\mathrm{Pic}^0(C)}(\Theta - [\mathcal{O}_C((g-1)p_0)])$; in particular we compute the Stiefel-Whitney class of the corresponding real line bundles on $\mathrm{Pic}^0(C)^{\hat{\iota}}$.

We have included two appendices for completeness: the first one contains a simple proof of the holonomy formula expressing the holonomy of a U(1)-connection along a contractible loop as the integral of the curvature. The second appendix is dedicated to a general $\mathbb{Z}_2$-localization formula which relates the Stiefel-Whitney numbers of a ι-Real bundle $(E, \tilde{\iota})$ on a compact manifold with involution (X, ι) to the Stiefel-Whitney classes of the real bundle $E^{\tilde{\iota}}$ over X^ι and the normal bundle of X^ι in X.

1. Families of Dirac operators of Klein surfaces

Families of Spin^c**-Dirac operators.** Let C be a compact Riemann surface of genus g. The spinor bundles of the *canonical* Spin^c-structure τ_{can} on C are $\Sigma^+_{\mathrm{can}} =$

Λ^0, $\Sigma^-_{\mathrm{can}} = \Lambda^{0,1}$, and the canonical Dirac operator of C associated with the canonical Spin^c-structure is (up to the factor $\sqrt{2}$) just the Dolbeault operator

$$\bar{\partial} : A^0 \longrightarrow A^{0,1} \ .$$

Using the canonical $H^2(C, \mathbb{Z})$-torsor structure on the set of equivalence classes of Spin^c-structures on C one obtains for any Hermitian line bundle L on C an L-twisted Spin^c-structure τ_L on C whose spinor bundles are $\Sigma^+_L := \Lambda^0(L)$, $\Sigma^-_L := \Lambda^{0,1}(L)$. For the construction of a Dirac operator for τ_L one needs a semi-connection δ on L, and then the corresponding Spin^c-Dirac operator will be

$$\delta : A^0(L) \to A^{0,1}(L) \ .$$

Varying δ in the space $\mathcal{A}^{0,1}(L)$ of semi-connections on L one gets a tautological family of elliptic operators parameterized by $\mathcal{A}^{0,1}(L)$.

Let $\mathcal{G}^{\mathbb{C}} := \mathcal{C}^{\infty}(C, \mathbb{C}^*)$ be the complex gauge group, which acts on $\mathcal{A}^{0,1}(L)$ from the right by $(\delta, g) \mapsto g^{-1} \circ \delta \circ g = \delta + g^{-1}\bar{\partial}g$. We are interested in descending this tautological family to the moduli space

$$\mathrm{Pic}^d(C) \simeq \mathcal{A}^{0,1}(L) \big/_{\mathcal{G}^{\mathbb{C}}}$$

of holomorphic structures on L. The problem here is that the group $\mathcal{G}^{\mathbb{C}}$ does not act freely on the affine space $\mathcal{A}^{0,1}(L)$, so the trivial line bundles

$$\mathcal{A}^{0,1}(L) \times A^0(L) \ , \ \mathcal{A}^{0,1}(L) \times A^{0,1}(L)$$

do not descend to $\mathrm{Pic}^d(C)$ in a natural way. In order to descend these bundles we choose a base point $p_0 \in C$ and consider the reduced complex gauge group

$$\mathcal{G}^{\mathbb{C}}_{p_0} := \{g \in \mathcal{G}^{\mathbb{C}}|\ g(p_0) = 1\} \ .$$

Since this group actes freely on $\mathcal{A}^{0,1}(L)$ we get Fréchet vector bundles

$$\mathscr{E}^0_{p_0} := \mathcal{A}^{0,1}(L) \times_{\mathcal{G}^{\mathbb{C}}_{p_0}} A^0(L) \ , \ \mathscr{E}^1_{p_0} := \mathcal{A}^{0,1}(L) \times_{\mathcal{G}^{\mathbb{C}}_{p_0}} A^{0,1}(L)$$

over $\mathcal{A}^{0,1}(L) \big/_{\mathcal{G}^{\mathbb{C}}}$, and a universal family of Dolbeault operators

$$\delta^L_{p_0} : \mathscr{E}^0_{p_0} \to \mathscr{E}^1_{p_0}$$

of index $d + 1 - g$ parameterized by $\mathrm{Pic}^d(C)$.

The determinant line bundle $\det \mathrm{ind}\ \delta^L_{p_0}$ has been extensively studied in the literature [18], [4], [6], [7], [8]. This line bundle has a natural holomorphic structure which can be described as follows. Consider the Poincaré line bundle

$$\mathbb{L}_{p_0} := \mathcal{A}^{0,1}(L) \times L \big/_{\mathcal{G}^{\mathbb{C}}_{p_0}}$$

over $\mathrm{Pic}^d(C) \times C$. Then one has a canonical isomorphism

$$\det \mathrm{ind}\ \delta^L_{p_0} \simeq \det(R^0 q_*(\mathbb{L}_{p_0})) \otimes \left[\det(R^1 q_*(\mathbb{L}_{p_0}))\right]^{\vee} \ ,$$

where q stands for the canonical projection $\mathrm{Pic}^d(C) \times C \to \mathrm{Pic}^d(C)$ (see [12], [6], [7], [8]), so $\det \mathrm{ind}\ \delta^L_{p_0}$ has a natural holomorphic structure. Choosing a different base point p_1 yields a new Poincaré line bundle $\mathbb{L}_{p_1}$, and the two Poincaré line bundles are related by a formula of the form

$$\mathbb{L}_{p_1} = \mathbb{L}_{p_0} \otimes q^*(\mathcal{M}) \ ,$$

where $\mathcal{M}$ is topologically trivial holomorphic line bundle on $\mathrm{Pic}^d(C)$. Using the projection formula for the functors $R^i q_*$ one obtains

$$\det \mathrm{ind}\, \delta_{p_1}^L \simeq \det \mathrm{ind}\, \delta_{p_0}^L \otimes \mathcal{M}^{\otimes(d+1-g)} \ .$$

This shows in particular that $\det \mathrm{ind}\, \delta_{p_0}^L$ is independent of the choice of p_0 when $d = g - 1$; in this case $\det \mathrm{ind}\, \delta_{p_0}^L$ has a canonical section whose zero locus is the geometric theta divisor $\Theta \subset \mathrm{Pic}^{g-1}(C)$, hence one has

$$(1) \qquad\qquad \det \mathrm{ind}\, \delta_{p_0}^L = \mathcal{O}_{\mathrm{Pic}^{g-1}(C)}(\Theta) \ ,$$

independently of p_0.

It is easy to compare to determinant line bundles $\det \mathrm{ind}\, \delta_{p_0}^L$, $\det \mathrm{ind}\, \delta_{p_0}^{L'}$ associated to differentiable line bundles L, L' of degrees d, d'. Let P_0 the underlying differentiable line bundle of $\mathcal{O}_C(p_0)$, and choose $L := L' \otimes P_0^k$, where $k = d - d'$. For L' we have a Poincaré line bundle

$$\mathbb{L}'_{p_0} := {}^{\mathcal{A}^{0,1}(L') \times L'} \big/ {}_{\mathcal{G}_{p_0}^{\mathbb{C}}} \ .$$

We denote by δ_{p_0} the semi-connection on P_0 defining the holomorphic structure of $\mathcal{O}_C(p_0)$, and we define the isomorphism

$$\varphi_{p_0} : \mathrm{Pic}^{d'}(C) \to \mathrm{Pic}^d(C)$$

by $\varphi_{p_0}([\delta']) := [\delta' \otimes \delta_{p_0}^{\otimes k}]$. One can easily check that

$$[\varphi_{p_0} \times \mathrm{id}_C]^*(\mathbb{L}_{p_0}) \simeq \mathbb{L}'_{p_0} \otimes p^*(\mathcal{O}_C(kp_0)) \ ,$$

where $p : \mathrm{Pic}^d(C) \times C \to C$ denotes the canonical projection. Note that

$$p^*(\mathcal{O}_C(kp_0)) \simeq \mathcal{O}_{\mathrm{Pic}^d(C) \times C}(k\Sigma_{p_0}) \ ,$$

where $\Sigma_{p_0} := \mathrm{Pic}^d(C) \times \{p_0\}$. When $k \geq 0$ we tensor the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathrm{Pic}^d(C) \times C} \longrightarrow \mathcal{O}_{\mathrm{Pic}^d(C) \times C}(k\Sigma_{p_0}) \longrightarrow \mathcal{O}_{k\Sigma_{p_0}}(k\Sigma_{p_0}) \longrightarrow 0$$

with $\mathbb{L}'_{p_0}$ and we write the corresponding long direct image exact sequence:

$$0 \to R^0 q_*(\mathbb{L}'_{p_0}) \to R^0 q_*([\varphi_{p_0} \times \mathrm{id}_C]^* \mathbb{L}_{p_0}) \to \mathrm{Pic}^d(C) \times \mathcal{O}_{kp_0} \to$$

$$\to R^1 q_*(\mathbb{L}'_{p_0}) \to R^1 q_*([\varphi_{p_0} \times \mathrm{id}_C]^* \mathbb{L}_{p_0}) \to 0 \ .$$

Lemma 1.1. *Let L be a differentiable line bundle of degree d on C and p_0 a base point. One has*

$$\det \mathrm{ind}\, \delta_{p_0}^L \simeq \mathcal{O}_{\mathrm{Pic}^d(C)}(\Theta + [\mathcal{O}_C((d - g + 1)p_0)]) \ .$$

Proof. Using the functoriality of the functor $\det$ and the exact sequence above we get

$$\varphi_{p_0}^*(\det \mathrm{ind}\, \delta_{p_0}^L) \simeq \det \mathrm{ind}\, \delta_{p_0}^{L'} \ .$$

In the special case $d' = g - 1$ this yields

$$\varphi_{p_0}^*(\det \mathrm{ind}\, \delta_{p_0}^L) \simeq \mathcal{O}_{\mathrm{Pic}^{g-1}(C)}(\Theta) \ .$$

If we denote by $\Theta + [\mathcal{O}_C((d-g+1)p_0)]$ the translate of the geometric Theta divisor $\Theta \subset \mathrm{Pic}^{g-1}(C)$ by the point $[\mathcal{O}_C((d - g + 1)p_0)] \in \mathrm{Pic}(C)$, the last isomorphism can be rewritten as $\det \mathrm{ind}\, \delta_{p_0}^L \simeq \mathcal{O}_{\mathrm{Pic}^d(C)}(\Theta + [\mathcal{O}_C((d-g+1)p_0)])$. ∎

In order to understand the line bundle $\det \operatorname{ind} \delta_{p_0}^L$ explicitly, we shall identify $\operatorname{Pic}^d(C)$ with the torus $\operatorname{Pic}^0(C)$ (which has an explicit description as the quotient $H^1(C, \mathcal{O})/H^1(X, \mathbb{Z})$) using the isomorphism $\otimes \mathcal{O}_C(dp_0) : \operatorname{Pic}^0(C) \to \operatorname{Pic}^d(C)$. We get

Lemma 1.2. *Via the identification* $\otimes \mathcal{O}_C(dp_0) : \operatorname{Pic}^0(C) \to \operatorname{Pic}^d(C)$ *the line bundle* $\det \operatorname{ind} \delta_{p_0}^L$ *on* $\operatorname{Pic}^d(C)$ *corresponds to the line bundle* $\mathcal{O}_{\operatorname{Pic}^0(C)}(\Theta - [\mathcal{O}_C((g-1)p_0)])$ *on* $\operatorname{Pic}^0(C)$.

In these constructions we considered families of Spin^c-Dirac operator obtained by coupling the canonical Spin^c-Dirac operator with holomorphic line bundles.

Families of Dirac operators associated with theta characteristics. A different point of view begins with the Dirac operator associated with a fixed Spin-structure on C. A Riemann surface of genus g has 2^{2g} equivalence classes of Spin-structures; these classes correspond bijectively to isomorphism classes of theta characteristics, i.e. of square roots κ of the canonical line bundle $\mathcal{K}_C$ [3].

The spinor bundles corresponding to a theta characteristic κ are

$$S^+ = \kappa \ , \ \ S^- = \Lambda^{0,1}(\kappa) \simeq \kappa^\vee \ ,$$

and the corresponding Dirac operator is (up to the factor $\sqrt{2}$) just the Dolbeault operator $\bar{\partial}_\kappa : A^0(\kappa) \to A^{0,1}(\kappa)$. A second natural way to construct a family of Dirac operators is to consider perturbations of this Spin-Dirac operator by flat line bundles.

To any form $\eta \in A^{0,1}$ we have a perturbed Dirac operator

$$\bar{\partial}_\kappa + \eta : A^0(\kappa) \to A^{0,1}(\kappa) \ .$$

Factorizing again by the reduced gauge group $\mathcal{G}_{p_0}^{\mathbb{C}}$ we obtain Fréchet bundles

$$\mathscr{F}_{p_0}^0 := A^{0,1} \times_{\mathcal{G}_{p_0}^{\mathbb{C}}} A^0(\kappa) \ , \ \ \mathscr{F}_{p_0}^1 := A^{0,1} \times_{\mathcal{G}_{p_0}^{\mathbb{C}}} A^{0,1}(\kappa)$$

over the quotient

$$A^{0,1}\big/_{\mathcal{G}_{p_0}^{\mathbb{C}}} \simeq A^{0,1}\big/_{\{g^{-1}\bar{\partial}g| \ g \in \mathcal{G}^{\mathbb{C}}\}} \simeq \operatorname{Pic}^0(C) \ ,$$

and a family of operators

$$\bar{\partial}_{\kappa,p_0} : \mathscr{F}_{p_0}^0 \to \mathscr{F}_{p_0}^1$$

parameterized by $\operatorname{Pic}^0(C)$. Note that the family $\bar{\partial}_{\kappa,p_0}$ is just the pull-back of the family $\delta_{p_0}^L : \mathscr{E}_{p_0}^0 \to \mathscr{E}_{p_0}^1$ via the isomorphism $[\mathcal{L}_0] \to [\mathcal{L}_0 \otimes \kappa]$, where L is the underlying differentiable line bundle of κ. This proves

Lemma 1.3. *Let* κ *be a theta characteristic on* C, p_0 *a base point. One has*

$$\det \operatorname{index} \bar{\partial}_{\kappa,p_0} \simeq \mathcal{O}_{\operatorname{Pic}^0(C)}(\Theta - [\kappa]) \ .$$

Real determinant line bundles and theta bundles on Klein surfaces. Let (C, ι) be a Klein surface with $C^\iota \neq \emptyset$. The Real structure ι induces a natural Real structure $\hat{\iota}$ on $\mathrm{Pic}(C)$ mapping $[\mathcal{L}]$ to $[\iota^*(\bar{\mathcal{L}})]$, which preserves each component $\mathrm{Pic}^d(C)$.

The involution $\hat{\iota}$ can be constructed with gauge theoretical methods in the following way. Fix a ι-Real [2] line bundle $(L, \tilde{\iota})$ of degree d (one can take for instance the underlying differentiable line bundle of the line bundle associated with a ι-invariant divisor of degree d). By definition this means that $\tilde{\iota}$ is a differentiable fibrewise antilinear, involution of L lifting ι. This involution induces anti-linear involutions $\tilde{\iota}^*$ on the spaces of L-valued forms $A^{0,q}(L)$ acting by

$$\tilde{\iota}^*(\sigma)(x) := \tilde{\iota}(\sigma(\iota(x))) \ , \ \tilde{\iota}^*(\alpha \otimes \sigma) := \overline{\iota^*(\alpha)} \otimes \tilde{\iota}^*(\sigma) \ \forall \alpha \in A^{0,q}(L) \ \forall \sigma \in \Gamma(L) \ .$$

For a semi-connection $\delta \in \mathcal{A}^{0,1}(L)$ put $\tilde{\iota}^*(\delta) := \tilde{\iota}^* \circ \delta \circ \tilde{\iota}^*$. Using the identity

$$\tilde{\iota}^*(\delta \cdot g) = \tilde{\iota}^*(\delta) \cdot \overline{\iota^*}(g)$$

we see that the map $\delta \mapsto \tilde{\iota}^*(\delta)$ induces an involution $\mathcal{A}^{0,1}(L)/\mathcal{G}^{\mathbb{C}} \to \mathcal{A}^{0,1}(L)/\mathcal{G}^{\mathbb{C}}$. Via the identification $\mathcal{A}^{0,1}(L)/\mathcal{G}^{\mathbb{C}} = \mathrm{Pic}^d(C)$ this involution coincides with $\hat{\iota}$, so it is independent of the choice of the ι-Real structure $\tilde{\iota}$ on L. Taking $p_0 \in C^\iota$ we see that the map $g \mapsto \overline{\iota^*}(g)$ leaves the subgroup $\mathcal{G}^{\mathbb{C}}_{p_0} \subset \mathcal{G}^{\mathbb{C}}$ invariant, and the product map

$$\tilde{\iota}^* \times \tilde{\iota} : \mathcal{A}^{0,1}(L) \times L \longrightarrow \mathcal{A}^{0,1}(L) \times L$$

induces an anti-holomorphic $(\hat{\iota} \times \iota)$-Real structure on the Poincaré line bundle $\mathbb{L}_{p_0}$ on $\mathrm{Pic}^d(C) \times C$. Regarding $\hat{\iota} \times \iota$ as a biholomorphism $\overline{\mathrm{Pic}}^d(C) \times \overline{C} \to \mathrm{Pic}^d(C) \times C$ and using the functoriality of $\det(R^0(\cdot)) \otimes [\det(R^1(\cdot))]^\vee$ with respect to biholomorphic base-change, we obtain

Remark 1.4. *Choosing $p_0 \in C^\iota$ we get a $\hat{\iota}$-Real structure on the determinant line bundle $\det$ index $\delta^L_{p_0}$ which is anti-holomorphic with respect to its natural holomorphic structure.*

This Real structure can be explicitly described fibrewise using the fibre identifications $\det$ index $\delta^L_{p_0}([\delta]) = \wedge^{\max} H^0(\mathcal{L}_\delta) \otimes \wedge^{\max} H^1(\mathcal{L}_\delta)^\vee$: it is induced by the anti-linear isomorphisms

$$H^0(\mathcal{L}_\delta) \to H^0(\mathcal{L}_{\tilde{\iota}^*(\delta)}) \ , \ H^1(\mathcal{L}_\delta) \to H^1(\mathcal{L}_{\tilde{\iota}^*(\delta)})$$

given by the operators $\tilde{\iota}^*$ on the spaces $A^0(L)$ and $A^{0,1}(L)$. Here we denoted by $\mathcal{L}_\delta$ the holomorphic line bundle defined by the semi-connection $\delta \in \mathcal{A}^{0,1}(L)$.

Note that the geometric theta divisor Θ is invariant under $\hat{\iota}$, so $\mathcal{O}_{\mathrm{Pic}^{g-1}(C)}(\Theta)$ has an obvious anti-holomorphic $\hat{\iota}$-Real structure. The same holds for the line bundle $\mathcal{O}_{\mathrm{Pic}^0(C)}(\Theta - [\mathcal{O}_C((g-1)p_0)])$ on $\mathrm{Pic}^0(C)$ when $p_0 \in C^\iota$. It is easy to see that, in general, two anti-holomorphic Real structures on the same holomorphic line bundle are congruent modulo S^1; therefore, using the isomorphism (1) and Lemma 1.2 we obtain

Remark 1.5. *For $p_0 \in C^\iota$ we have isomorphisms of $\hat{\iota}$-Real line bundles*

$$\det \mathrm{ind} \ \delta^L_{p_0} \simeq \mathcal{O}_{\mathrm{Pic}^{g-1}(C)}(\Theta) \ ,$$

$$\{\otimes \mathcal{O}_C(dp_0)\}^* (\det \mathrm{ind} \ \delta^L_{p_0}) \simeq \mathcal{O}_{\mathrm{Pic}^0(C)}(\Theta - [\mathcal{O}_C((g-1)p_0)]).$$

Fix now a ι-Real theta-characteristic $(\kappa, \tilde\iota_\kappa)$, i.e. a square root κ of $\mathcal{K}_C$ endowed with an anti-holomorphic ι-Real structure $\tilde\iota_\kappa$. We will see (see Proposition 2.12) that the set of isomorphism classes of such pairs $(\kappa, \tilde\iota_\kappa)$ corresponds bijectively to the finite subset of $\mathrm{Pic}^{g-1}(C)$ consisting of $\hat\iota$-invariant square roots of $[\mathcal{K}_C]$.

Remark 1.6. *Choosing $p_0 \in C^\iota$, and a ι-Real theta-characteristic $(\kappa, \tilde\iota_\kappa)$ we get a $\hat\iota$-Real structure on the determinant line bundle* $\det \mathrm{index}\, \bar\partial_{\kappa,p_0}$ *and an isomorphism*

$$\det \mathrm{index}\, \bar\partial_{\kappa,p_0} \simeq \mathcal{O}_{\mathrm{Pic}^0(C)}(\Theta - [\kappa])$$

of $\hat\iota$-Real line bundles on $\mathrm{Pic}^0(C)$.

As we explained in the introduction, our first goal is to identify the differentiable underlying line bundles of the $\hat\iota$-Real determinant line bundles

$$\{\otimes \mathcal{O}_C(dp_0)\}^* \, (\det \mathrm{ind}\, \delta^L_{p_0}) \simeq \mathcal{O}_{\mathrm{Pic}^0(C)}(\Theta - [\mathcal{O}_C((g-1)p_0)])$$

$$\det \mathrm{index}\, \bar\partial_{\kappa,p_0} \simeq \mathcal{O}_{\mathrm{Pic}^0(C)}(\Theta - [\kappa])$$

on $(\mathrm{Pic}^0(C), \hat\iota)$ as elements in the cohomology group $H^1_{\mathbb{Z}_2}(\mathrm{Pic}^0(C), \underline{S}^1(1))$, and in particular to compute the Stiefel-Whitney class of the associated fixed point real line bundles over $\mathrm{Pic}^0(C)^{\hat\iota}$. Note that $\mathrm{Pic}^0(C)^{\hat\iota}$ is a disjoint union of a finite family of real sub-tori of $\mathrm{Pic}^0(C)$ parameterized by the quotient $H^1(C, \mathbb{Z})^{\iota^*}/(\mathrm{id} + \iota^*) H^1(C, \mathbb{Z})$ (see section 2.2).

Orientability of the components of $S^d(C)^\iota$. The symmetric power $S^d(C)$ can be regarded as a projective fibration over $\mathrm{Pic}^d(C)$ via the map $\lambda : S^d(C) \to \mathrm{Pic}^d(C)$ defined by $\lambda(D) := [\mathcal{O}(D)]$. The fibre over $[\mathcal{L}] \in \mathrm{Pic}^d(C)$ is the projective space $\mathbb{P}(H^0(\mathcal{L}))$. Suppose now for simplicity that $d > 2(g-1)$, so that $h^1(\mathcal{L}) = 0$ and $h^0(\mathcal{L}) = d + 1 - g$ for every holomorphic line bundle of degree d. We can use the Poincaré line bundle $\mathbb{L}_{p_0}$ on $\mathrm{Pic}^d(C) \times C$ and identify $S^d(C)$ with the projectivization $\mathbb{P}(\mathcal{E}_0)$ of the holomorphic bundle $\mathcal{E}_0$ on $\mathrm{Pic}^d(C)$ which is associated with the locally free sheaf $R^0 q_*(\mathbb{L}_{p_0})$. This bundle comes with an $\hat\iota$-Real structure $\hat\iota_0$ induced by the family of anti-linear isomorphisms $H^0(\mathcal{L}_\delta) \to H^0(\mathcal{L}_{\tilde\iota^*(\delta)})$ defined by the map

$$\tilde\iota^* : A^0(L) \to A^0(L) \,, \ \tilde\iota^*(\sigma)(x) := \tilde\iota(\sigma(\iota(x))) \,.$$

It is easy to see that the involution $S^d(C) \to S^d(C)$ induced by ι coincides with $\mathbb{P}(\hat\iota_0) : \mathbb{P}(E_0) \to \mathbb{P}(E_0)$ via the canonical identification $S^d(C) \simeq \mathbb{P}(R^0 q_*(\mathbb{L}_{p_0}))$. The fixed point bundle $F_0 := \mathcal{E}_0^{\hat\iota_0}$ is a real $(d + 1 - g)$-bundle over $\mathrm{Pic}^d(C)^{\hat\iota}$, and one obtains a natural identification $S^d(C)^\iota \simeq \mathbb{P}_{\mathbb{R}}(F_0)$. The relative Euler sequence associated with the projective fibre bundle $q_0 : \mathbb{P}_{\mathbb{R}}(F_0) \to \mathrm{Pic}^d(C)^{\hat\iota}$ reads

$$0 \to \mathbb{P}_{\mathbb{R}}(F_0) \times \mathbb{R} \to q_0^*(F_0)(\lambda^\vee) \to T_{q_0} \to 0 \,.$$

Here $T_{q_0} \subset T_{\mathbb{P}_{\mathbb{R}}(F_0)}$ stands for the vertical tangent subbundle of q_0, and λ denotes the tautological line bundle on $\mathbb{P}_{\mathbb{R}}(F_0)$. Note that all connected components of the base $\mathrm{Pic}^d(C)^{\hat\iota}$ are tori, so they are orientable. Therefore

$$(2) \quad w_1(T_{\mathbb{P}_{\mathbb{R}}(F_0)}) = w_1(T_{q_0}) = w_1(q_0^*(F_0)(1)) = q_0^*(w_1(F_0)) + (d + 1 - g)w_1(\lambda^\vee) \,.$$

On the other hand

$$w_1(F_0) = w_1(\wedge^{d+1-g}(F_0)) = w_1((\det \mathrm{ind}\, \delta^L_{p_0})^{\tilde{\hat\iota}_d}) \,,$$

where $\tilde{\hat\iota}_d$ denotes the canonical $\hat\iota$-Real structure on the determinant line bundle $\det \mathrm{ind}\, \delta^L_{p_0}$, given by Remark 1.4. It is convenient to take the pull-back of all

the considered objects via the map $\{\otimes \mathcal{O}_C(dp_0)\}^*$ to $\mathrm{Pic}^0(C)$, which is an abelian variety coming with an explicit description as quotient of a vector space by a lattice. Using formula (2), Remark 1.5 and denoting by $\tilde{\iota}$ the canonical $\hat{\iota}$-Real structure on $\mathcal{O}_{\mathrm{Pic}^0(C)}\left(\Theta - [\mathcal{O}_C((g-1)p_0)]\right)$, we obtain:

Proposition 1.7. *Let (C, ι) be a Klein surface, $p_0 \in C^\iota$, and let $d > 2(g-1)$. Regard $S^d(C)^\iota$ as a bundle over $\mathrm{Pic}^0(C)^{\hat{\iota}}$ via the map $D \mapsto [\mathcal{O}_C(D - dp_0)]$. Let $T \subset \mathrm{Pic}^0(C)^{\hat{\iota}}$ be a connected component of $\mathrm{Pic}^0(C)^{\hat{\iota}}$ and $S^d(C)^\iota_T$ the corresponding component of $S^d(C)^\iota$. Then $S^d(C)^\iota_T$ is orientable if and only if $d + 1 - g$ is even and*

$$w_1\left(\mathcal{O}_{\mathrm{Pic}^0(C)}\left(\Theta - [\mathcal{O}_C((g-1)p_0)]\right)^{\tilde{\iota}}\Big|_T\right) = 0 \ .$$

Note that the last condition depends effectively on the component T (see Proposition 3.8, section 3.2). This gives a clear geometric motivation for the computation of the first Stiefel-Whitney classes $w_1\left(\mathcal{O}_{\mathrm{Pic}^0(C)}\left(\Theta - [\mathcal{O}_C((g-1)p_0)]\right)^{\tilde{\iota}}\Big|_T\right)$ corresponding to the components $T \in \pi_0(\mathrm{Pic}^0(C)^{\hat{\iota}})$.

2. REAL HERMITIAN LINE BUNDLES

2.1. **Grothendieck's formalism.** Let X be a paracompact space endowed with an involution ι. Regard X as a $\mathbb{Z}_2$-space, and denote by $\underline{S}$ (respectively $\underline{S}(1)$) the $\mathbb{Z}_2$-sheaf on X of S^1-valued smooth functions, with the $\mathbb{Z}_2$-action defined by composition with ι (respectively by composition with ι and conjugation).

We recall from [10] the following classification theorem for equivariant principal bundles.

Proposition 2.1. *Let (X, α), $\alpha : \Gamma \times X \to X$ be a paracompact Γ-space, where Γ is a finite group, and let G be a Lie group endowed with a group morphism $\alpha : \Gamma \to \mathrm{Aut}(G)$. Then there is a canonical bijection*

$$\{\text{Iso classes of } \alpha\text{-equivariant principal } G\text{-bundles}\} \simeq H^1_\Gamma(X; \underline{G}(\alpha)) \ ,$$

where $\underline{G}(\alpha)$ stands for the Γ-sheaf of continuous G-valued maps on X with Γ-action defined via α.

Remark 2.2. *When X is a differentiable manifold one obtains a similar result replacing the Γ-sheaf of continuous G-valued maps by the Γ-sheaf of smooth G-valued maps on X. However the cohomology sets associated with the two sheaves can be identified as in the non-equivariant case.*

This can be seen be comparing the standard spectral exact sequences associated with the two sheaves at the E_2-level.

For any Abelian group A one has two obvious $\mathbb{Z}_2$-actions on A: the trivial action α_0 and the inversion action α_1. We agree to write (0) and (1) for the twistings by α_0 and α_1, and we agree to omit (0). Let $^-$ be the conjugation action of $\mathbb{Z}_2$ on $\mathbb{C}$ and $\mathbb{C}^*$.

Remark 2.3. *Let (X, ι) be a space with involution. The set of isomorphism classes of ι-Real line bundles on X can be identified with the set of isomorphism classes of Hermitian ι-Real line bundles on X. More precisely the monomorphism $\underline{S}^1(1) = \underline{S}^1(^-) \to \underline{\mathbb{C}}^*(^-)$ induces isomorphism in positive cohomology.*

Indeed, it suffices to see that $H^k_{\mathbb{Z}_2}(X, \underline{\mathbb{R}}) = 0$ for any $k > 0$ (using again the standard spectral sequence associated with this sheaf).

In particular, we obtain an identification

$$\{[L, \tilde{\iota}] \mid (L, \tilde{\iota}) \text{ Real line bundle over } (X, \iota)\} = H^1_{\mathbb{Z}_2}(X, \underline{S}^1(1)) \ .$$

Using the standard spectral exact sequence associated with the $\mathbb{Z}_2$-sheaf $\underline{S}^1(1)$, and denoting by $\mathcal{G}(1)$ the $\mathbb{Z}_2$-module structure on the gauge group $\mathcal{G}$ defined by the involution $g \mapsto \iota^*(\bar{g})$, one obtains an exact sequence

$$E_2^{1,0} = H^1_{\mathbb{Z}_2}(H^0(X, \underline{S}^1(1))) \longrightarrow H^1_{\mathbb{Z}_2}(X, \underline{S}^1(1)) \longrightarrow E_2^{0,1} = H^0_{\mathbb{Z}_2}(H^1(X, \underline{S}^1(1))) =$$

$$(3) \qquad = H^0_{\mathbb{Z}_2}(H^2(X, \mathbb{Z})(1)) \xrightarrow{d_2} E_2^{2,0} = H^2_{\mathbb{Z}_2}(H^0(X, \underline{S}^1(1))) \ ,$$

The $\mathbb{Z}_2$-module $H^0(X, \underline{S}^1(1))$ in the first and in the last term above is just the gauge group $\mathcal{G}$ regarded as a $\mathbb{Z}_2$-module via the involution $g \mapsto \iota^*(\bar{g})$). This $\mathbb{Z}_2$-module will be denoted by $\mathcal{G}(1)$, and will keep the notation $\mathcal{G}$ for the $\mathbb{Z}_2$-module structure defined by the involution $g \mapsto \iota^*(g)$ (not the trivial $\mathbb{Z}_2$-module structure!); the first cohomology group $H^1_{\mathbb{Z}_2}(\mathcal{G}(1))$ of $\mathcal{G}(1)$ fits into the short exact sequence of $\mathbb{Z}_2$-modules

$$0 \longrightarrow \mathcal{G}(1)^{\mathbb{Z}_2} \longrightarrow \mathcal{G} \xrightarrow{\Sigma} \mathcal{G}^{\mathbb{Z}_2} \longrightarrow H^1_{\mathbb{Z}_2}(\mathcal{G}(1)) \longrightarrow 0 \ ,$$

where $\Sigma : \mathcal{G} \to \mathcal{G}^{\mathbb{Z}_2}$ is the morphism $g \mapsto g\iota^*(g)$. This proves the following

Proposition 2.4. *One has an exact sequence*

$$1 \to \mathcal{G}(1)^{\mathbb{Z}_2} \to \mathcal{G} \xrightarrow{\Sigma} \mathcal{G}^{\mathbb{Z}_2} \xrightarrow{\lambda} H^1_{\mathbb{Z}_2}(X, \underline{S}^1(1)) \xrightarrow{c_1} H^2(X, \mathbb{Z})(1)^{\mathbb{Z}_2} \xrightarrow{o} H^2_{\mathbb{Z}_2}(\mathcal{G}(1)),$$

where

(1) *The morphism Σ is given by $\Sigma(g) := g(\iota^*g)$,*

(2) $\ker(c_1) = {\mathcal{G}^{\mathbb{Z}_2}}\big/{\Sigma(\mathcal{G})} = H^1_{\mathbb{Z}_2}(\mathcal{G}(1))$,

(3) $H^2(X, \mathbb{Z})(1)^{\mathbb{Z}_2} = \{x \in H^2(M, \mathbb{Z}) \mid \iota^*(x) = -x\}$.

To compute $H^1_{\mathbb{Z}_2}(\mathcal{G}(1))$ we use the short exact sequence of $\mathbb{Z}_2$-modules

$$(4) \qquad 0 \longrightarrow \mathcal{G}_0(1) \longrightarrow \mathcal{G}(1) \longrightarrow H^1(X, \mathbb{Z})(1) \longrightarrow 0 \ ,$$

where $\mathcal{G}_0 = \mathcal{C}^\infty(X, \mathbb{R})/\mathbb{Z}$ is the connected component of the identity in $\mathcal{G}$. This connected component fits into the short exact sequence

$$(5) \qquad 0 \longrightarrow \mathbb{Z}(1) \longrightarrow \mathcal{C}^\infty(X, \mathbb{R})(1) \longrightarrow \mathcal{G}_0(1) \longrightarrow 0 \ .$$

One has $H^k_{\mathbb{Z}_2}(\mathcal{C}^\infty(X, \mathbb{R})(1)) = 0$ for all $k \geq 1$. Therefore

$$H^{2k-1}_{\mathbb{Z}_2}(\mathcal{G}_0(1)) = H^{2k}_{\mathbb{Z}_2}(\mathbb{Z}(1)) = 0 \ , \ H^{2k}_{\mathbb{Z}_2}(\mathcal{G}_0(1)) = H^{2k+1}_{\mathbb{Z}_2}(\mathbb{Z}(1)) = \mathbb{Z}_2 \ , \ \forall k \geq 1 \ .$$

We get an exact sequence

$$0 \to H^1_{\mathbb{Z}_2}(\mathcal{G}(1)) \to H^1_{\mathbb{Z}_2}(H^1(X, \mathbb{Z})(1)) \to H^2_{\mathbb{Z}_2}(\mathcal{G}_0(1)) \to H^2_{\mathbb{Z}_2}(\mathcal{G}(1)) \to$$

$$\to H^2_{\mathbb{Z}_2}(H^1(X, \mathbb{Z})(1)) \to 0 \ .$$

When $X^\iota \neq \emptyset$, we choose $x_0 \in X^\iota$ and we notice that the composition of the maps

$$S^1(1) \longrightarrow \mathcal{G}_0(1) \longrightarrow \mathcal{G}(1) \xrightarrow{\mathrm{ev}_{x_0}} S^1(1)$$

is the identity, and that the first map $S^1(1) \longrightarrow \mathcal{G}_0(1)$ induces an isomorphism in cohomology of strictly positive degree. This is so since $S^1(1)$ fits in the short exact sequence $1 \to \mathbb{Z}(1) \to \mathbb{R}(1) \to S^1(1) \to 1$, which can be easily compared to (5). Therefore the morphism $H^2_{\mathbb{Z}_2}(\mathcal{G}_0(1)) \to H^2_{\mathbb{Z}_2}(\mathcal{G}(1))$ is injective, and we get

Remark 2.5. *When $X^\iota \neq \emptyset$, the natural map*

$$H^1_{\mathbb{Z}_2}(\mathcal{G}(1)) \longrightarrow H^1_{\mathbb{Z}_2}(H^1(X, \mathbb{Z})(1))$$

is an isomorphism and one has the short exact sequence

$$0 \to H^2_{\mathbb{Z}_2}(\mathcal{G}_0(1)) \simeq \mathbb{Z}_2 \xrightarrow{j} H^2_{\mathbb{Z}_2}(\mathcal{G}(1)) \xrightarrow{q} H^2_{\mathbb{Z}_2}(H^1(X, \mathbb{Z})(1)) \longrightarrow 1 .$$

The generator of

$$H^2_{\mathbb{Z}_2}(\mathcal{G}_0(1)) = \mathcal{G}_0(1)^{\mathbb{Z}_2} \big/ \{g\iota^*(\bar{g}) |\ g \in \mathcal{G}_0\}$$

is the class modulo $\{g\iota^*(\bar{g}) |\ g \in \mathcal{G}_0\}$ of the constant gauge transformation $-1 \in S^1$.

Lemma 2.6. *Suppose $X^\iota \neq \emptyset$. The morphism*

$$d_2 : E_2^{0,2} = H^0_{\mathbb{Z}_2}(H^2(X, \mathbb{Z})(1)) = H^2(X, \mathbb{Z})(1)^{\mathbb{Z}_2} \longrightarrow E_2^{2,0} = H^2_{\mathbb{Z}_2}(\mathcal{G}(1))$$

has the property $\ker(d_2) = \ker(q \circ d_2)$.

Proof. Indeed, if $c \in H^2(X, \mathbb{Z})(1)^{\mathbb{Z}_2}$ belongs to $\ker(q \circ d_2)$, then we get $d_2(c) \in j(H^2_{\mathbb{Z}_2}(\mathcal{G}_0(1)))$. It suffices to notice that $d_2(c)$ can never coincide with the class $[-1]$ modulo $\{g\iota^*(\bar{g}) |\ g \in \mathcal{G}_0\}$. This can be seen as follows:

The morphism $d_2 : H^2(X, \mathbb{Z})(1)^{\mathbb{Z}_2} \to H^2_{\mathbb{Z}_2}(\mathcal{G}(1))$ can be geometrically interpreted as follows: consider a Hermitian line bundle L on X with Chern class $c_1(L) = c \in H^2(X, \mathbb{Z})(1)^{\mathbb{Z}_2}$. Since $\iota^*(c) = -c$, it follows that $\iota^*(L) \simeq \bar{L}$, so there exists an anti-linear isometry $\sigma : L \to \iota^*(L)$. We get a smooth family of anti-linear isometries $\sigma_x : L_x \to L_{\iota(x)}$. The composition $\phi_x = \sigma_{\iota_x} \circ \sigma_x : L_x \to L_x$ is $\mathbb{C}$-linear, so it can be regarded as an element in S^1, depending smoothly on $x \in X$. It is easy to see that $\iota^*(\phi) = \bar{\phi}$. The element $d_2(c)$ is just the class $[\phi]$ modulo the subgroup $\{g\iota^*(\bar{g}) |\ g \in \mathcal{G}_0\}$. We have to show that $[\phi] \neq [-1]$. Choose $x_0 \in X^\iota$ and a unitary identification $L_{x_0} \simeq \mathbb{C}$. The anti-linear isometry σ_{x_0} acts as $\sigma_{x_0}(l) = \zeta \bar{l}$, for a constant $\zeta \in S^1$. Therefore $\phi_{x_0} = \zeta \bar{\zeta} = 1$. If $[\phi] = [-1]$, one would have $\phi_{x_0} = -\psi_{x_0} \bar{\psi}_{x_0} = -1$ for a smooth S^1-valued function ψ, which yields obviously a contradiction. ∎

Using Proposition 2.4 and Remark 2.5 we obtain:

Corollary 2.7. *Suppose $X^\iota \neq \emptyset$. There exists an exact sequence*

$$0 \to H^1_{\mathbb{Z}_2}(H^1(X, \mathbb{Z})(1)) \to H^1_{\mathbb{Z}_2}(X, \underline{S}^1(1)) \xrightarrow{c_1} H^2(X, \mathbb{Z})(1)^{\mathbb{Z}_2} \xrightarrow{\mathcal{O}} H^2_{\mathbb{Z}_2}(H^1(X, \mathbb{Z})(1)) .$$

As in the classical classification theory for vector bundles, is important to give an explicit description of the set $H^1_{\mathbb{Z}_2}(X, \underline{S}^1(1))$ of isomorphism classes of ι-Real line bundles on X in terms of characteristic classes. The relevant characteristic classes associated to a ι-Real Hermitian line bundle $(L, \tilde{\iota})$ on (X, ι) are:

$$c_1(L) \in \ker(\mathcal{O}) \subset H^2(X, \mathbb{Z})(1)^{\mathbb{Z}_2} ,\ w_1(L^{\tilde{\iota}}) \in H^1(X^\tau, \mathbb{Z}_2) .$$

Therefore is a natural problem to determine explicitly the kernel and the image of the corresponding group morphism

$$H^1_{\mathbb{Z}_2}(X, \underline{S}^1(1)) \xrightarrow{cw} H^2(X, \mathbb{Z})(1)^{\mathbb{Z}_2} \times H^1(X^\tau, \mathbb{Z}_2) .$$

Determining these groups will give an alternative short exact sequence having the group $H^1_{\mathbb{Z}_2}(X, \underline{S}^1(1))$ as central term.

2.2. **Examples.** In this section we will apply the general formalism developed above in two important cases: a Klein surface and a Real torus.

The case of a Klein surface. Let C be a closed connected, oriented differentiable 2-manifold, and $\iota : C \to C$ an orienting reversing involution with $C^\iota \neq \emptyset$. Let $r \in \mathbb{N}$ be the number of components of C^ι. Let $d_2 : H^2(C, \mathbb{Z}) \to \mathbb{Z}_2$, $\deg_{\mathbb{Z}_2} : H^1(C^\iota, \mathbb{Z}_2) \to \mathbb{Z}_2$ be the morphisms defined by

$$\alpha \mapsto \langle \alpha, [C] \rangle \bmod 2 \ , \ \gamma \mapsto \langle \gamma, [C^\iota]_{\mathbb{Z}_2} \rangle$$

and denote by $H^2(C, \mathbb{Z})(1)^{\mathbb{Z}_2} \times_{\mathbb{Z}_2} H^1(C^\iota, \mathbb{Z}_2)$ the fibre product of d_2 and $\deg_{\mathbb{Z}_2}$.

Theorem 2.8. *The characteristic map*

$$cw : H^1_{\mathbb{Z}_2}(C, \underline{S}^1(1)) \longrightarrow H^2(C, \mathbb{Z})(1)^{\mathbb{Z}_2} \times H^1(C^\iota, \mathbb{Z}_2)$$

induces an isomorphism

$$H^1_{\mathbb{Z}_2}(C, \underline{S}^1(1)) \xrightarrow{cw} H^2(C, \mathbb{Z})(1)^{\mathbb{Z}_2} \times_{\mathbb{Z}_2} H^1(C^\iota, \mathbb{Z}_2) \ .$$

Proof. It follows from the Corollary in Appendix B that for any ι-Real line bundle $(L, \tilde{\iota})$ one has

$$\deg_{\mathbb{Z}_2}(w_1(L^{\tilde{\iota}})) \equiv= \langle c_1(L), [C] \rangle \ (\bmod\ 2) \ \ .$$

This shows that $\mathrm{im}(cw) \subset H^2(C, \mathbb{Z}) \times_{\mathbb{Z}_2} H^1(C^\tau, \mathbb{Z}_2)$, and that we have a commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \to & H^1_{\mathbb{Z}_2}(H^1(C, \mathbb{Z})(1)) & \to & H^1_{\mathbb{Z}_2}(C, \underline{S}^1(1)) & \xrightarrow{c_1} & H^2(C, \mathbb{Z}) & \to & 0 \\
& & \downarrow \mathrm{j} & & \downarrow cw & & \| & & \\
0 & \to & \ker(\deg_{\mathbb{Z}_2}) & \to & H^2(C, \mathbb{Z}) \times_{\mathbb{Z}_2} H^1(C^\iota, \mathbb{Z}_2) & \xrightarrow{\mathrm{pr}_1} & H^2(C, \mathbb{Z}) & \to & 0
\end{array}
$$

One easily sees that cw is surjective e.g. by choosing a ι-anti-invariant holomorphic structure J on C and using Real divisors to construct real line bundles with prescribed characteristic classes.

Using a Comessatti basis for the $\mathbb{Z}_2$-module $H^1(C, \mathbb{Z})$ one computes

$$H^1_{\mathbb{Z}_2}(H^1(C, \mathbb{Z})(1)) \simeq \mathbb{Z}_2^{r-1} \ .$$

On the other hand one obviously has $\ker(\deg_{\mathbb{Z}_2}) \simeq \mathbb{Z}_2^{r-1}$. The claim follows now from the snake lemma. ∎

The case of a torus. Let now be $T = V/\Lambda$ a torus, where V is a real vector space, and $\Lambda \subset V$ is a lattice. Let $\tau : \Lambda \to \Lambda$ be a linear involution, and denote by the same symbol the induced automorphisms of V and T.

In order to describe the fixed point locus T^τ we use the short exact sequence of $\mathbb{Z}_2$-modules

$$0 \longrightarrow \Lambda \longrightarrow V \longrightarrow T \longrightarrow 0 \ .$$

The corresponding long exact sequence of cohomology groups reads:

$$0 \longrightarrow \Lambda^\tau \longrightarrow V^\tau \longrightarrow T^\tau \longrightarrow H^1_{\mathbb{Z}_2}(\Lambda) = {}^{\Lambda^{-\tau}}\!\big/\!_{(\mathrm{id} - \tau)\Lambda} \longrightarrow 0 \ .$$

This shows that T^τ decomposes as a disjoint union

$$T^\tau = \bigcup_{[\mu] \in H^1_{\mathbb{Z}_2}(\Lambda)} T_{[\mu]} \ ,$$

where every connected component T_κ is a torus isomorphic to the quotient

$$T_0 := {}^{V^\tau}\!/_{\Lambda^\tau} \ ,$$

so $H_1(T_{[\mu]}, \mathbb{Z}) = \Lambda^\tau$, $H^1(T_{[\mu]}, \mathbb{Z}) = [\Lambda^\tau]^\vee$.

We are interested in the τ-Real line bundles on T. Since $H^1(T, \mathbb{Z}) = \Lambda^\vee$, one gets the exact sequence

$$0 \to H^1_{\mathbb{Z}_2}(\Lambda^\vee) = {}^{[\Lambda^\vee]^{\tau^*}}\!/_{(\mathrm{id} + \tau^*)\Lambda^\vee} \to H^1_{\mathbb{Z}_2}(T, \underline{S}^1(1)) \xrightarrow{c_1} H^2(T, \mathbb{Z})(1)^{\mathbb{Z}_2} =$$

$$= \wedge^2 \Lambda^\vee(1)^{\mathbb{Z}_2} \xrightarrow{\mathcal{O}} H^2_{\mathbb{Z}_2}(\Lambda^\vee(1)) \simeq H^1_{\mathbb{Z}_2}(\Lambda^\vee) = {}^{[\Lambda^\vee]^{-\tau^*}}\!/_{(\mathrm{id} - \tau^*)\Lambda^\vee} \ .$$

We will see that on a torus the obstruction map $\mathcal{O}$ vanishes (see Proposition 3.5). For every τ-Real line bundle $(L, \tilde\tau)$ on T, we have an associated Stiefel-Whitney class of the fixed point bundle $L^{\tilde\iota}$ on T^τ, which is an element $w(L, \tilde\tau) \in H^1(T^\tau, \mathbb{Z}_2)$. Such an element can be regarded as a map

$$w(L, \tilde\tau) : {}^{\Lambda^{-\tau}}\!/_{(\mathrm{id} - \tau)\Lambda} \to \mathrm{Hom}(\Lambda^\tau, \mathbb{Z}_2) \ .$$

In section 3 we will give an explicit description of the group $H^1_{\mathbb{Z}_2}(T, \underline{S}^1(1))$ of isomorphism classes of τ-Real Hermitian line bundles in terms of characteristic classes. We will show that the group morphism

$$H^1_{\mathbb{Z}_2}(T, \underline{S}^1(1)) \xrightarrow{cw} H^2(T, \mathbb{Z})(1)^{\mathbb{Z}_2} \times H^1(T^\tau, \mathbb{Z}_2) =$$

$$= \wedge^2 \Lambda^\vee(1)^{\mathbb{Z}_2} \times \mathrm{Map}\left({}^{\Lambda^{-\tau}}\!/_{(\mathrm{id} - \tau)\Lambda} \ , \ \mathrm{Hom}(\Lambda^\tau, \mathbb{Z}_2)\right)$$

is injective, and we will determine explicitly its image. In particular we will describe the set of maps

$$ {}^{\Lambda^{-\tau}}\!/_{(\mathrm{id} - \tau)\Lambda} \longrightarrow \mathrm{Hom}(\Lambda^\tau, \mathbb{Z}_2)$$

which correspond to τ-Real line bundles on T i.e. which have the form $w(L, \tilde\tau)$ for a τ-Real line bundle $(L, \tilde\tau)$ on T.

2.3. Real line bundles and connections.

Proposition 2.9. *Let (X, ι) be a differentiable manifold endowed with an involution. Let L be a Hermitian line bundle on X whose Chern class satisfies $\iota^*(c_1(L)) = -c_1(L)$, and let $\mathcal{B}(L)$ be the moduli space of Hermitian connections on L. Then*

(1) *ι induces a well defined involution $\hat\iota : \mathcal{B}(L) \to \mathcal{B}(L)$.*
(2) *Suppose $X^\iota \neq \emptyset$. The following conditions are equivalent:*
   (a) *$\mathcal{B}(L)^{\hat\iota} \neq \emptyset$.*
   (b) *L admits ι-Real structures.*
(3) *If one of the two equivalent conditions above is satisfied, then the set of isomorphism classes of ι-Real structures on L can be identified with $\pi_0(\mathcal{B}(L)^{\hat\iota})$.*

Proof. (1) We denote by $c : S^1 \to S^1$ the conjugation automorphism. Fix an ι-covering anti-isomorphism $f : L \to L$, and denote by the same symbol the induced ι-covering type c-isomorphism $P_L \to P_L$ between associated principal bundles.

For a connection A on P_L we define

$$\hat{\iota}([A]) := [f^*(A)] \ ,$$

where $f^*(A)$ is the pull-back connection in the sense of [KN]. In terms of connection forms one has

$$\theta_{f^*(A)} = -f^*(\theta_A) \ .$$

Since f is well defined up to composition which a gauge transformation, it follows that $\hat{\iota}$ is well-defined. Since $f \circ f$ is a gauge transformation, it follows that $\hat{\iota}$ is an involution as claimed.

(2), (3) Let $\mathcal{R}$ be the space of ι-Real structures on L. The gauge group $\mathcal{G}$ acts on $\mathcal{R}$ by conjugation, and the set of isomorphism classes of ι-Real structures on L is the quotient $\mathcal{R}/\mathcal{G}$.

In order to prove (2) and (3) it suffices to construct a surjective map

$$F : \mathcal{B}(L)^{\hat{\iota}} \to \mathcal{R}/\mathcal{G}$$

whose fibers are the connected components $\mathcal{B}(L)^{\hat{\iota}}$. Let $A \in \mathcal{A}(L)$ such that $[A] \in \mathcal{B}^{\hat{\iota}}$. It follows that there exists a gauge transformation $g \in \mathcal{G}$ such that $A = g^* f^*(A)$, in other words $A = \tilde{\iota}_A^*(A)$ where $\tilde{\iota}_A := f \circ g$ is a type-c ι-covering isomorphism. Note that $\tilde{\iota}_A$ is well defined up to multiplication with constant elements $\zeta \in S^1$. The composition $\tilde{\iota}_A \circ \tilde{\iota}_A$ is an A-parallel gauge transformation, so it is a constant gauge transformation $z \mathrm{id}_L$. Let $x \in X^\iota$ and $v \in L_x$. One has

$$(\tilde{\iota}_A \circ \tilde{\iota}_A \circ \tilde{\iota}_A)(v) = \tilde{\iota}_A((\tilde{\iota}_A \circ \tilde{\iota}_A)(v)) = \tilde{\iota}_A(zv) = \bar{z}\tilde{\iota}_A(v) = (\tilde{\iota}_A \circ \tilde{\iota}_A)(\tilde{\iota}_A(v)) = z\tilde{\iota}_A(v) \ ,$$

so $z \in \mathbb{R} \cap S^1 = \{\pm 1\}$. But L_x is a complex line, so it does not admit any anti-linear automorphism with square $-\mathrm{id}_{L_x}$. This shows that $z = 1$, so $\tilde{\iota}_A$ is an involution. Since it is also ι-covering and anti-linear, we get $\tilde{\iota}_A \in \mathcal{R}$. Replacing A by a gauge equivalent connection produces an ι-Real structure which is conjugate to $\tilde{\iota}_A$, so we get a well defined element $F([A]) := [\tilde{\iota}_A]$.

To see that the map F is surjective note that a ι-Real structure $\tilde{\iota}$ defines an involution on the affine space $\mathcal{A}(L)$. But any involution on an affine space has fixed points. For an $\tilde{\iota}$-invariant connection A one gets obviously $F([A]) = [\tilde{\iota}]$.

It is easy to see that F is continuous with respect to the quotient topologies. Indeed, the ι-real structure $\tilde{\iota}$ associated with A is $\tilde{\iota} = f \circ g$, where

$$g^{-1}dg = A - f^*(A) \ ,$$

which shows that the class $[g] \in \mathcal{G}/S^1$ depends continuously on $A \in \mathcal{A}(L)$. On the other hand the equivalence class $[f \circ g] \in \mathcal{R}/\mathcal{G}$ depends only on $[g] \in \mathcal{G}/S^1$, so it depends continuously on A as claimed.

Since the quotient topology on $\mathcal{R}/\mathcal{G}$ is discrete, F is constant on the connected components of $\mathcal{B}(L)^{\hat{\iota}}$.

It remains to prove that the fibers of F are connected. Let $A, B \in \mathcal{A}(L)$ such that $F([A]) = F([B])$. It follows that there exists $g, h, k \in \mathcal{G}$ such that

$$A = (f \circ g)^*(A) \ , \ B = (f \circ h)^*(B) \ , \ f \circ g = k \circ (f \circ h) \circ k^{-1} \ ,$$

which implies $k^*(A) = (f \circ h)^* \circ k^*(A)$. Therefore the connections $k^*(A) \sim A$ and B are both $\tilde{\iota}$-invariant, where $\tilde{\iota}$ is the ι-Real structure $f \circ h$. But is easy to see that

the space of $\tilde{\iota}$-invariant connections in $\mathcal{A}(L)$ is an affine subspace with model linear space $iA^1(X,\mathbb{R})^{-\iota^*}$, so this space is connected as claimed. ∎

Suppose now that X was endowed with a ι-Riemannian metric g, and is compact. A statement similar to the one above holds when one replaces the infinite dimensional space $\mathcal{B}(L)$ with the moduli space $\mathcal{T}(L)$ of Yang-Mills connections on L, which is a $b_1(X)$-dimensional torus. Note that the inclusion map

$$\mathcal{T}(L) \hookrightarrow \mathcal{B}(L)$$

is a homotopy equivalence.

Corollary 2.10. *Let (X, ι) be a compact Riemannian manifold endowed with an isometric involution. Let L be a Hermitian line bundle on X whose Chern class satisfies $\iota^*(c_1(L)) = -c_1(L)$, and let $\mathcal{T}(L)$ be the moduli space of Yang-Mills connections on L. Then*

(1) *ι induces a well defined involution $\hat{\iota} : \mathcal{T}(L) \to \mathcal{T}(L)$.*
(2) *Suppose $X^\iota \neq \emptyset$. The following conditions are equivalent:*
   (a) *$\mathcal{T}(L)^{\hat{\iota}} \neq \emptyset$.*
   (b) *L admits ι-Real structures.*
(3) *If one of the two equivalent conditions above is satisfied, then the set of isomorphism classes of ι-Real structures on L can be identified with $\pi_0(\mathcal{T}(L)^{\hat{\iota}})$.*

It is useful to consider the disjoint union of all Yang-Mills tori

$$\mathcal{T}_X := \coprod_{c \in H^2(X,\mathbb{Z})} \mathcal{T}(L_c) \ ,$$

where L_c denotes a Hermitian line bundle with Chern class c. This union comes with a well defined involution $\hat{\iota}$ defined as the composition of the usual pull-back of connections:

$$\mathcal{A}(L_c) \ni A \mapsto \iota^*(A) \in \mathcal{A}(\iota^*(L_c))$$

with the canonical identification $\mathcal{A}(L) = \mathcal{A}(\bar{L})$ induced by the equality between the total spaces of the principal bundles P_L, $P_{\bar{L}}$. Using Corollary 2.10 we obtain

Proposition 2.11. *Under the conditions and with the notations of Corollary 2.10 the assignment $A \mapsto [\tilde{\iota}_A]$ defines a group morphism*

$$F_X : \mathcal{T}_X^{\hat{\iota}} \longrightarrow H^1_{\mathbb{Z}_2}(X, \underline{S}^1(1))$$

from the fixed point locus $\mathcal{T}_X^{\hat{\iota}}$ to the group of isomorphism classes of ι-Real Hermitian line bundles on X. This morphism induces an isomorphism

$$f_X : \pi_0(\mathcal{T}_X^{\hat{\iota}}) \xrightarrow{\simeq} H^1_{\mathbb{Z}_2}(X, \underline{S}^1(1)) \ .$$

This result has an important complex geometric analogon:

Proposition 2.12. *Let X be a compact connected complex manifold endowed with an anti-holomorphic involution $\iota : X \to X$. Suppose that $X^\iota \neq \emptyset$. Consider the induced anti-holomorphic involution $\hat{\iota} : \operatorname{Pic}(X) \to \operatorname{Pic}(X)$ defined by*

$$\hat{\iota}([\mathcal{L}]) := [\iota^*(\overline{\mathcal{L}})] \ .$$

Let $\mathcal{L}$ be a holomorphic line bundle on X with $[\mathcal{L}] \in \operatorname{Pic}(X)^{\hat{\iota}}$. Then

(1) *There exists an anti-holomorphic ι-Real structure $\tilde{\iota}_{\mathcal{L}}$ on $\mathcal{L}$, which is unique up to multiplication with constant elements $\zeta \in S^1$.*

(2) *The assignment* $[\mathcal{L}] \mapsto [(\mathcal{L}, \tilde{\iota}_{\mathcal{L}})]$ *defines a group morphism*

$$\mathfrak{F}_X : \mathrm{Pic}(X)^{\hat{\iota}} \longrightarrow H^1_{\mathbb{Z}_2}(X, \underline{S}^1(1))$$

which maps the fixed point locus $\mathrm{Pic}(X)^{\hat{\iota}}$ *to the set of isomorphism classes of* ι-*Real Hermitian line bundles.*

(3) *The induced map* $\mathfrak{f}_X : \pi_0(\mathrm{Pic}(X)^{\hat{\iota}}) \to H^1_{\mathbb{Z}_2}(X, \underline{S}^1(1))$ *defines a bijection between* $\pi_0(\mathrm{Pic}(X)^{\hat{\iota}})$ *and the set of isomorphism classes of* ι-*Real line bundles with Chern class in the Neron-Severi group* $\mathrm{NS}(X)$ *of* X.

Proof. (1), (2) The construction of the ι-Real structure $\tilde{\iota}_{\mathcal{L}}$ is similar to the construction of the ι-Real structure $\tilde{\iota}_A$ in Proposition 2.9.

(3) If X is Kählerian, the statement follows from Corollary 2.11. Indeed, on Kählerian manifolds the space of Hermite-Einstein connections coincides with the space of integrable Yang-Mills connections. Therefore, using the Kobayashi-Hitchin correspondence for line bundles [14], we see that $\mathrm{Pic}(X)$ can be identified with the (open and closed) subgroup of the Yang-Mills group $\mathcal{T}_X$ consisting of gauge equivalence classes of Yang-Mills connections on line bundles with Chern class of type (1,1).

For the non-Kählerian case, one has to replace the Yang-Mills group $\mathcal{T}_X$ with the group $\mathcal{T}_X^{HE}$ of gauge-equivalence classes of Hermite-Einstein connections, and to see that the analogue of Corollary 2.11 holds, giving a bijection

$$\pi_0(\mathrm{Pic}(X)^{\hat{\iota}}) = \pi_0((\mathcal{T}_X^{HE})^{\hat{\iota}}) \to \{\gamma \in H^1_{\mathbb{Z}_2}(\underline{S}^1(1)) \mid c_1(\gamma) \in \mathrm{NS}(X)\} .$$

$\blacksquare$

3. Real line bundles on a torus

3.1. Abelian Yang-Mills theory on a torus. Let $T = V/\Lambda$ be a n-dimensional torus, where V is an n-dimensional real vector space of dimension n and $\Lambda \subset V$ a rank n lattice such that $\langle \Lambda \rangle_{\mathbb{R}} = V$. Let $u \in \mathrm{Alt}^2(\Lambda, \mathbb{Z}) = H^2(T, \mathbb{Z})$ be an alternated $\mathbb{Z}$-valued form on Λ; we will denote by the same symbol the corresponding differential form $u \in A^2(V)$ on V, and by $\bar{u}$ the differential form on T whose pull-back via the projection $p : V \to T$ is u; $\bar{u}$ is just the harmonic representative of the 2-cohomology class $u \in H^2(T, \mathbb{Z})$ with respect to any flat metric on T induced by an inner product on V.

Let L be a Hermitian line bundle of Chern class u on T, and $\mathcal{T}(L)$ the torus of Yang-Mills connections on L.

Our first goals are:

(1) Describe explicitly the torus $\mathcal{T}(L)$ of Yang-Mills connections on L,
(2) For every Yang-Mills class $[A]$ describe the holonomy with respect to A along the loops of the form $p[v_0, v_0 + \lambda] \subset T$, $v_0 \in V$, $\lambda \in \Lambda$.

Let A be any Hermitian connection on L. We define a map $\alpha^A : \Lambda \to S^1$ by the condition

$$h^A_{c_\lambda}(\zeta) = \alpha^A_\lambda \zeta ,$$

where c_λ is the loop (based in the origin $0_T \in T$) defined by $c_\lambda(t) := p(\lambda t)$, and h^A stands for the holonomy associated with the connection A. The loops $c_{\lambda'} * c_\lambda$

and $c_{\lambda+\lambda'}$ are homotopic. Using the homotopy formula and supposing that λ, λ' are linearly independent over $\mathbb{R}$, one obtains

$$(6) \qquad (\alpha^A_{\lambda+\lambda'})^{-1}\alpha^A_{\lambda'}\alpha^A_{\lambda} = e^{\int_{T(\lambda,\lambda')} p^*(F_A)} \ ,$$

where $T(\lambda,\lambda') \subset V$ is the triangle given by the convex hull of the points 0_V, λ, λ' oriented in the obvious way.

Suppose now that A is Yang-Mills with respect to the flat metric on T induced by any inner product on V. The Yang-Mills condition means that $\frac{i}{2\pi}F_A$ is the harmonic representative of the Chern class $c_1(L) \in H^2(T,\mathbb{Z})$. We denote by u the corresponding element in $\mathrm{Alt}^2(\Lambda,\mathbb{Z})$ and we agree to use the same symbol for the corresponding constant differential form $u \in A^2(V)$ on V. With this notation, the Yang-Mills condition becomes

$$p^*(\frac{i}{2\pi}F_A) = u \ ,$$

so the holonomy identities (6) become

$$\alpha^A_{\lambda+\lambda'} = \alpha^A_{\lambda}\alpha^A_{\lambda'}e^{2\pi i \int_{T(\lambda,\lambda')} u} = \alpha^A_{\lambda}\alpha^A_{\lambda'}e^{\pi i u(\lambda,\lambda')} \ .$$

Definition 3.1. *A map $\alpha : \Lambda \to S^1$ is called u-character if the following identity holds:*

$$\alpha_{\lambda+\lambda'} = \alpha_{\lambda}\alpha_{\lambda'}e^{\pi i u(\lambda,\lambda')} \ \ \text{for } \lambda, \ \lambda' \in \Lambda.$$

Note that (if non-empty) the set $\mathrm{Hom}_u(\Lambda, S^1)$ of u-characters is a $\mathrm{Hom}(\Lambda, S^1)$-torsor, so it has a natural differentiable structure which makes it (non-canonically) diffeomorphic to the dual torus $V^\vee/\Lambda^\vee$.

Proposition 3.2. $\mathrm{Hom}_u(\Lambda, S^1)$ *is non-empty and the assignment $A \mapsto \alpha^A$ defines a canonical isomorphism*

$$h : \mathcal{T}(L) \to \mathrm{Hom}_u(\Lambda, S^1)$$

of differentiable $\mathrm{Hom}(\Lambda, S^1)$-torsors.

Proof: For any Yang-Mills connection A the corresponding map α^A is a u-character, so the first statement is clear. Note first that the tensor product of connections defines a natural $\mathcal{T}_0$-torsor structure on $\mathcal{T}(L)$, where $\mathcal{T}_0$ stands for the torus of flat Yang-Mills connections on T. But $\mathcal{T}_0$ can be identified with $\mathrm{Hom}(\pi_1(T,0_T), S^1) = \mathrm{Hom}(\Lambda, S^1)$ via the holonomy map h by a classical result in differential geometry. It suffices to notice that tensor product of abelian connections corresponds to multiplication of holonomies, so h defines a morphism of $\mathrm{Hom}(\Lambda, S^1)$-torsors. ∎

One can prove directly that the set of u-characters is non-empty in the following way:

Remark 3.3. *One can prove directly that the set of u-characters is non-empty:*
Choose a basis $(e_1,\ldots,e_n)$ of Λ. Then any system $z = (z_i)_{1\le i\le n}$ of elements in S^1 defines a unique u-character α with the property $\alpha(e_i) = z_i$.

Our next goal is to describe explicitly the Yang-Mills connection which corresponds to a given u-character α, and in particular to compute the holonomy associated to this connection along more general loops.

Let $(e_1,\ldots,e_n)$ be a $\mathbb{Z}$-basis of Λ (so a $\mathbb{R}$-basis of V), and let $(x^1,\ldots,x^n)$ be the dual basis of $V^\vee$. Any x_i will be regarded as a smooth function on V.

Putting $u_{jk} := u(e_j, e_k)$ one has

$$u = \frac{1}{2} \sum_{j,k} u_{jk} dx^j \wedge dx^k = \sum_{j<k} u_{jk} dx^j \wedge dx^k \ .$$

Let v be the tangent field on V given by

$$\mathrm{v}_v = v \in T_v(V) = V \quad \forall v \in V$$

We define the imaginary differential 1-form θ_u on V by

$$\theta_u := -\pi i \iota_{\mathrm{v}} u = -\pi i \sum_{j,k} u_{jk} x^j dx^k \ .$$

One has $d\theta_u = -2\pi i u$, which shows that θ_u is the connection form of a connection A_u on the trivial Hermitian line bundle $V \times \mathbb{C}$ with curvature $F_{A_u} = -2\pi i u$ and Chern form $c_1(A_u) = \frac{i}{2\pi} F_{A_u} = u$. The covariant derivative corresponding to A_u is given by the formula $\nabla_u := d + \theta_u$.

In order to compute the holonomy of A_u along a segment $[v_0, v_0 + w] \subset V$, we parameterize this segment by $c_{v_0, w} : [0, 1] \to V$, $c_{v_0, w}(t) := v_0 + tw$. The covariant derivative of the pull-back connection $c^*(A_u)$ on the trivial line bundle $[0, 1] \times \mathbb{C}$ over $[0, 1]$ is:

$$d + c^*(\theta_u) = d - \pi i \sum_{j,k} u_{jk}(v_0 + tw)^j w^k dt = d - \pi i \sum_{j,k} u_{jk} v_0^j w^k dt = d - \pi i u(v_0, w) \ .$$

Therefore the parallel transport in the trivial line bundle $V \times \mathbb{C}$ along c with respect to A_u and with the initial condition $\zeta(0) = 1$ is defined by the Cauchy problem:

$$\dot{\zeta} - \pi i u(v_0, w)\zeta = 0 \ , \ \zeta(0) = 1 \ .$$

This has the obvious solution $\zeta(t) = e^{\pi i u(v_0, w)t}$. Therefore

Remark 3.4. *The holonomy*

$$h_{c_{v_0,w}}^{A_u} : \{v_0\} \times \mathbb{C} \to \{v_0 + w\} \times \mathbb{C}$$

of the connection A_u along $c_{v_0,w}$ is given by $h_{c_{v_0,w}}^{A_u}(\zeta) = e^{\pi i u(v_0, w)}\zeta$.

The action of the lattice Λ on V can be lifted to an action on the trivial line bundle $V \times \mathbb{C}$ by choosing a factor of automorphy $e = (e_\lambda)_{\lambda \in \Lambda}$, i.e. a system of functions $e_\lambda : V \to S^1$ satisfying the identities:

(7) $$e_{\lambda'}(v + \lambda)e_\lambda(v) = e_{\lambda + \lambda'}(v) \ \forall \ \lambda, \ \lambda' \in \Lambda \ .$$

The Λ-action $\tilde{e}$ on $V \times \mathbb{C}$ corresponding to a factor of automorphy e is given by

$$\lambda \cdot (v, z) = \tilde{e}_\lambda(v, z) := (v + \lambda, e_\lambda(v)z) \ .$$

Denote by $L(e)$ the line bundle on T obtained as the Λ-quotient of the trivial line bundle $V \times \mathbb{C}$ with respect to $\tilde{e}$. We seek factors of automorphy e such that the connection A_u descends to a connection $A(e)$ on the quotient line bundle $L(e)$. This is equivalent to the condition that A_u is $\tilde{e}$-invariant.

Let s_0 be the constant section $s_0(v) := (v, 1)$ in $V \times \mathbb{C}$, and denote by $T_\lambda : V \to V$ the translation defined by λ. The condition $\tilde{e}_\lambda^*(A_u) = A_u$ is equivalent with:

$$\nabla_u((\tilde{e}_\lambda)_*(s_0)) = (\tilde{e}_\lambda)_*(\nabla_u(s_0)) \ .$$

Note that $(\tilde{e}_\lambda)_*(s_0) = (e_\lambda \circ T_{-\lambda})s_0$, hence the invariance condition becomes:

$$de_\lambda + e_\lambda T_\lambda^* \theta_u = e_\lambda \theta_u \ ,$$

or, equivalently,

$$d\log(e_\lambda) = \pi i \ \iota_{\underline{\lambda}} u = d(u(\lambda, \cdot)) \ .$$

Therefore, the factor of automorphy e must have the form

$$e_\lambda(v) = a_\lambda e^{\pi i u(\lambda, v)} \ ,$$

where a_λ is a constant. Since e must satisfy the cocycle condition (7) we see that the function $a : \Lambda \to S^1$ must satisfy the condition

$$a_{\lambda + \lambda'} = a_\lambda a_{\lambda'} e^{\pi i u(\lambda, \lambda')} \ ,$$

i.e. a is a u-character. Note that, by Remark 3.4, the holonomy of the connection A_u along the segment $[0, \lambda] \subset V$ is trivial. Taking into account that, in the construction of $L(e)$, the identification between the fibers $\{0\} \times \mathbb{C}$, $\{\lambda\} \times \mathbb{C}$ is defined by $e_\lambda(0)$, one sees that the u-character $\alpha^{A(e)}$ associated to the Yang-Mills connection $A(e)$ is given by $\alpha^{A(e)} = \bar{a}$. This gives a geometric interpretation of the factor a_λ appearing in the expression of e_λ.

3.2. **Real Yang-Mills connections on a torus.** Let $\tau : \Lambda \to \Lambda$ be an automorphism of order 2 of the lattice Λ; denote by the same symbol the induced involutions $V \to V$ and $T \to T$. Recall (see Corollary 2.7) that the group $H^1_{\mathbb{Z}_2}(T, \underline{S}^1(1))$ of isomorphism classes of Real Hermitian line bundles on T fits in the short exact sequence
(8)
$$0 \to {}^{[\Lambda^\vee]^{\tau^*}}\big/(\mathrm{id} + \tau^*)\Lambda^\vee \to H^1_{\mathbb{Z}_2}(T, \underline{S}^1(1)) \xrightarrow{c_1} \wedge^2 \Lambda^\vee(1)^{\mathbb{Z}_2} \xrightarrow{\mathcal{O}} {}^{[\Lambda^\vee]^{-\tau^*}}\big/(\mathrm{id} - \tau^*)\Lambda^\vee \ .$$

The obstruction space ${}^{[\Lambda^\vee]^{-\tau^*}}\big/(\mathrm{id} - \tau^*)\Lambda^\vee$ coincides with the cohomology group

$$H^1_{\mathbb{Z}_2}(\Lambda^\vee) = H^2_{\mathbb{Z}_2}(\mathrm{Hom}(\Lambda, S^1)) = {}^{\mathrm{Hom}(\Lambda, S^1)^\tau}\big/(\mathrm{id} \cdot \tau)\mathrm{Hom}(\Lambda, S^1) \ .$$

We know that a Hermitian line bundle L admits τ-Real structures if and only if the fixed point locus of the induced involution $\tau^* : \mathcal{T}(L) \to \mathcal{T}(L)$ is non-empty. Using the identification between $\mathcal{T}(L)$ and the space $\mathrm{Hom}_u(\Lambda, S^1)$ of u-characters the involution τ^* becomes:

$$\tau^*(\alpha)_\lambda = \bar{\alpha}_{\tau(\lambda)} \ .$$

The existence of a τ-Real structure on L is therefore equivalent to the existence of a u-character α satisfying the τ-Reality condition

(9)
$$\alpha_{\tau(\lambda)} = \bar{\alpha}_\lambda \ .$$

Using this remark one can compute explicitly the obstruction $\mathcal{O}(u)$ of a form $u \in \wedge^2 \Lambda^\vee(1)^{\mathbb{Z}_2}$ in the following way: Fix any u-character α, and consider the function $\rho_\alpha : \Lambda \to S^1$ given by $\lambda \mapsto \alpha(\lambda)\alpha(\tau(\lambda))$; since $\tau^*(u) = -u$, this function is a τ-invariant character, and its class modulo $(\mathrm{id} \cdot \tau)\mathrm{Hom}(\Lambda, S^1)$ is independent of α. This class is the obstruction $\mathcal{O}(u)$.

Proposition 3.5. *On a torus the obstruction map $\mathcal{O}$ vanishes.*

Proof: Choose a Comessatti basis $(\alpha_1, \ldots, \alpha_a, \beta_1, \ldots \beta_s, \gamma_{s+1}, \ldots, \gamma_{n-a})$ of (Λ, τ), i.e. a basis satisfying

$$\tau(\alpha_i) = \alpha_i \text{ for } 1 \le i \le a \ ; \ \tau(\beta_j) = \alpha_j - \beta_j \text{ for } 1 \le j \le s \ ;$$

(10) $$\tau(\gamma_k) = -\gamma_k \ \text{ for } s + 1 \le k \le n - a \ .$$

Here $s \le a$ is the Comessatti characteristic of (Λ, τ), and n is the rank of Λ (see Lemma 3.5 in [19]). As in Remark 3.3 we see that for any system

$$z = (u_1, \ldots, u_a, v_1, \ldots v_s, w_{s+1}, \ldots, w_{n-a})$$

of elements in S^1 we get a u-character α_z such that $\alpha(\alpha_i) = u_i$, $\alpha(\beta_j) = v_j$, $\alpha(\gamma_k) = w_k$. Note that the τ-Reality condition (9) holds if and only if it holds for the elements of the Comessatti basis (which is τ-invariant). Therefore α_z is τ-Real if and only if

$$u_i \in \{\pm 1\} \text{ for } 1 \le i \le a \ \text{ and } \ u_j = e^{\pi i u(\alpha_j, \beta_j)} \text{ for } 1 \le j \le s \ .$$

∎

Remark 3.6. *The proof of Proposition 3.5 shows that, in the presence of a Comessatti basis, the space of τ-Real u-characters can be explicitly identified with the space $\{\pm 1\}^{a-s} \times [S^1]^{n-a}$. The connected components of this space correspond bijectively to isomorphism classes of τ-Real line bundles.*

Our goal is now the following: Let $(L, \tilde{\tau})$ be a τ-Real Hermitian line bundle on T. The fixed point locus $L^{\tilde{\tau}}$ is a $\mathbb{R}$-line bundle on the fixed point locus T^τ, and we want to compute its Stiefel-Whitney class $w_1(L^{\tilde{\tau}})$. The fixed point locus T^τ is a disjoint union of components $T_{[\mu]}$ all translations of the torus V^τ / Λ^τ by elements $[\mu] \in \frac{1}{2}\Lambda^{-\tau} / \frac{1}{2}(\mathrm{id} - \tau)\Lambda$, so the Stiefel-Whitney class $w_1(L^{\tilde{\tau}})$ can be regarded as an function

$$w : {\frac{1}{2}\Lambda^{-\tau}} \Big/ {\frac{1}{2}(\mathrm{id} - \tau)\Lambda} \longrightarrow \mathrm{Hom}(\Lambda^\tau, \{\pm 1\}) \ .$$

For a class $[\mu]$ the morphism $w([\mu]) \in \mathrm{Hom}(\Lambda^\tau, \{\pm 1\})$ has a simple geometric interpretation: $w([\mu])(\lambda)$ is just the holonomy of any $\tilde{\tau}$-invariant Hermitian connection A on L along the path $p \circ c_{\mu, \lambda}$. This follows from the following general

Remark 3.7. *Let $F \to B$ be an Euclidean line bundle on a differentiable manifold B, and A its unique $\mathrm{O}(1)$-connection (which is automatically flat). Let $\gamma : S^1 \to B$ be a smooth map, and $h = \gamma_*([S^1]) \in H_1(B, \mathbb{Z})$. Then*

$$\langle w_1(F), h \rangle = h_\gamma^A \in \{\pm 1\} \simeq \mathbb{Z}_2 \ .$$

Suppose now that $L = L(e)$ where e is the factor of automorphy associated with a τ-Real u-character a. In this case one can use the Yang-Mills connection $A(e)$, which is τ-Real. Using Remark 3.4 we see that the holonomy along the closed path $p \circ c_{\mu, \lambda}$ is given by

$$h(\zeta) = e_\lambda(\mu)^{-1} e^{\pi i u(\mu, \lambda)} \zeta = \bar{a}_\lambda e^{-\pi i u(\lambda, \mu) + \pi i u(\mu, \lambda)} \zeta = \alpha_\lambda e^{\pi i u(2\mu, \lambda)} \zeta \ .$$

Therefore we get:

$$w([\mu])(\lambda) = \alpha_\lambda e^{\pi i u(2\mu, \lambda)} \ \forall \lambda \in \Lambda^\tau \ .$$

Note that $w(0)(\lambda) = \alpha_\lambda$, so that one obtains the transformation formula

$$w([\mu])(\lambda) = w(0)(\lambda)e^{\pi i u(2\mu,\lambda)} \ \forall \lambda \in \Lambda^\tau \ .$$

Identifying $H_1(T_{[\mu]}, \mathbb{Z})$ with Λ^τ and $H^1(T_{[\mu]}, \mathbb{Z}_2)$ with $\mathrm{Hom}(\Lambda^\tau, \mathbb{Z}_2)$ we obtain the following *difference formula*

Proposition 3.8. *One has*

$$(11) \qquad\qquad w([\mu]) - w(0) = u(2\mu, \cdot) \ (mod \ 2) \ .$$

This formula shows that the function w is completely determined by $w(0)$. Note also that the difference $w([\mu]) - w(0)$ vanishes on subgroup $(\mathrm{id} + \tau)\Lambda \subset \Lambda^\tau$ of trivial invariants in Λ. Indeed, one has

$$u(2\mu, \lambda + \tau(\lambda)) = u(2\mu, \lambda) + u(2\mu, \tau(\lambda)) = u(2\mu, \lambda) - u(2\tau(\mu), \lambda) = 2u(2\mu, \lambda) \in 2\mathbb{Z} \ .$$

It follows that the restriction $w([\mu])\big|_{(\mathrm{id}+\tau)\Lambda}$ is independent of $[\mu]$. We want to identify this restriction.

Note first that the map $\tilde{f}_u : \Lambda \to \mathbb{Z}_2$ defined by $\tilde{f}_u(\lambda) := u(\lambda, \tau(\lambda)) \ (\mathrm{mod} \ 2)$ is a group morphism. This morphism vanishes on the subgroup $\Lambda^{-\tau}$ of anti-invariant elements, because for any $\nu \in \Lambda^{-\tau}$ one has

$$u(\lambda + \nu, \tau(\lambda + \nu)) = u(\lambda, \tau(\lambda)) + u(\lambda, \tau(\nu)) + u(\nu, \tau(\lambda)) + u(\nu, \tau(\nu)) =$$

$$= u(\lambda, \tau(\lambda)) + 2u(\lambda, \tau(\nu)) \ .$$

Therefore the morphism $\tilde{f}_u$ induces a well-defined morphism $f_u : (\mathrm{id} + \tau)\Lambda \to \mathbb{Z}_2$ given by

$$f_u(\lambda + \tau(\lambda)) := \tilde{f}_u(\lambda) = u(\lambda, \tau(\lambda)) \ (mod \ 2).$$

Note also that the morphism $\mathrm{Alt}^2(\Lambda, \mathbb{Z}) \to \mathrm{Hom}((\mathrm{id} + \tau)\Lambda, \mathbb{Z}_2)$ given by $u \mapsto f_u$ is obviously a group morphism. Using the identification $\{\pm 1\} = \mathbb{Z}_2$ we have

Remark 3.9. *For any* $[\mu] \in \frac{1}{2}\Lambda^{-\tau}/\frac{1}{2}(1 - \tau)\Lambda$ *it holds* $w([\mu])\big|_{(\mathrm{id}+\tau)\Lambda} = f_u$.

Proof: The holonomy along the closed path $p \circ c_{\mu, \lambda + \tau(\lambda)}$ is given by

$$h(\zeta) = \alpha_{\lambda + \tau(\lambda)} = e^{\pi i u(2\mu, \lambda + \tau(\lambda))}\zeta \ .$$

Since $u(2\mu, \lambda + \tau(\lambda)) = u(2\mu, \lambda) + u(2\mu, \tau(\lambda)) = u(2\mu, \lambda) - u(2\tau(\mu), \lambda) = 2u(2\mu, \lambda) \in \mathbb{Z}$, one gets

$$w(\zeta)(\lambda + \tau(\lambda)) = \alpha_\lambda \bar{\alpha}_\lambda e^{\pi i u(\lambda, \tau(\lambda))} = e^{\pi i u(\lambda, \tau(\lambda))} \ .$$

$\blacksquare$

We can now summarize our results and give an explicit formula for the Stiefel-Whitney class $w_1(L^{\tilde{\iota}})$ of the fixed point locus $L^{\tilde{\iota}}$ of a τ-Real line bundle on a torus T. We can suppose that L is endowed with an invariant Yang-Mills connection $A = A(e)$, where e is the factor of automorphy associated with a τ-Real u-character $a = (a_\lambda)_{\lambda \in \Lambda}$:

Proposition 3.10. *For elements* $\lambda \in \Lambda^\tau$ *and* $[\mu] \in \frac{1}{2}\Lambda^{-\tau}/\frac{1}{2}(\mathrm{id} - \tau)\Lambda$ *it holds*

$$w([\mu])(\lambda) = \bar{a}_\lambda e^{\pi i u(2\mu, \lambda)} \ .$$

In particular, $w(0)(\lambda) = \bar{a}_\lambda$ *for every* $\lambda \in \Lambda^\tau$.

3.3. Classification of Real line bundles on a Real torus.

Let again $\tau : \Lambda \to \Lambda$ be an automorphism of order 2 of an n-dimensional lattice $\Lambda \subset V = \langle \Lambda \rangle_{\mathbb{R}}$, and denote by the same symbol the induced involutions on V and $T := V/\Lambda$.

Our next goal is a complete classification – in terms of characteristic classes – of the Grothendieck group $H^1_{\mathbb{Z}_2}(T, \underline{S}^1(1))$ of τ-Real line bundles on T.

For any $u \in \mathrm{Alt}^2(\Lambda, \mathbb{Z})^{-\tau^*}$ we put

$$W(u) := \{w \in \mathrm{Hom}(\Lambda^\tau, \mathbb{Z}_2)|\ w_{|(\mathrm{id}+\tau)\Lambda} = f_u\} \ .$$

Consider the fibre product

$$\mathrm{Alt}^2(\Lambda, \mathbb{Z})^{-\tau^*} \times_{\mathrm{Hom}((\mathrm{id}+\tau)\Lambda, \mathbb{Z}_2)} \mathrm{Hom}(\Lambda^\tau, \mathbb{Z}_2) \ ,$$

where $\mathrm{Alt}^2(\Lambda, \mathbb{Z})^{-\tau^*}$, $\mathrm{Hom}(\Lambda^\tau, \mathbb{Z}_2)$ are regarded as a groups over $\mathrm{Hom}((\mathrm{id}+\tau)\Lambda, \mathbb{Z}_2)$ via $u \mapsto f_u$, and $w \mapsto w_{|(\mathrm{id}+\tau)\Lambda}$ respectively.

By Remark 3.9 it follows that for every τ-Real line bundle $(L, \tilde{\tau})$, the Stiefel-Whitney class of the restriction $L^{\tilde{\tau}}|_{T_0}$ of the real line bundle $L^{\tilde{\tau}}$ to the standard connected component $T_0 := V^\tau/\Lambda^\tau$ of T^τ is an element of $W(u)$.

Theorem 3.11. *The group morphism*

$$cw_0 : H^1_{\mathbb{Z}_2}(T, \underline{S}^1(1)) \to \mathrm{Alt}^2(\Lambda, \mathbb{Z})^{-\tau^*} \times_{\mathrm{Hom}((\mathrm{id}+\tau)\Lambda, \mathbb{Z}_2)} \mathrm{Hom}(\Lambda^\tau, \mathbb{Z}_2)$$

defined by

$$cw_0(L, \tilde{\tau}) := (c_1(L), w_1(L^{\tilde{\tau}}|_{T_0})) \ ,$$

is a bijection.

Proof:

1. *Injectivity:* An element of $\ker(cw_0)$ is the class of a τ-Real line bundle $(L, \tilde{\tau})$ with trivial first Chern class $c_1(L)$ and vanishing Stiefel-Whitney class $w_1(L^{\tilde{\tau}}|_{T_0})$. This first condition implies that $(L, \tilde{\tau})$ is induced by an element

$$[\chi] \in {[\Lambda^\vee]^{\tau^*}}/{(\mathrm{id} + \tau^*)\Lambda^\vee} \ ,$$

i.e. it coincides with the flat τ-Real line bundle $(L_\chi, \tilde{\tau}_\chi)$, where L_χ is induced by the τ-invariant representation $e^{\pi i \chi} : \pi_1(T) \to \{\pm 1\}$ associated with the τ-invariant functional χ; L_χ can be endowed with a natural τ-Real structure $\tilde{\tau}_\chi$.

Note now that the natural morphism

$${[\Lambda^\vee]^{\tau^*}}/{(\mathrm{id} + \tau^*)\Lambda^\vee} \to \mathrm{Hom}(\Lambda^\tau, \mathbb{Z}_2)$$

is a monomorphism, and its image is

$$\mathrm{Hom}\left({\Lambda^\tau}/{(\mathrm{id} + \tau)\Lambda}, \mathbb{Z}_2\right) \subset \mathrm{Hom}(\Lambda^\tau, \mathbb{Z}_2) \ .$$

This can easily be proved using a Comessatti basis in Λ. Therefore, the vanishing of $w_1(L^{\tilde{\tau}}|_{T_0})$ implies $[\chi] = 0$.

2. *Surjectivity:* Let $(u, w) \in \mathrm{Alt}^2(\Lambda, \mathbb{Z})^{-\tau^*} \times_{\mathrm{Hom}((\mathrm{id}+\tau)\Lambda, \mathbb{Z}_2)} \mathrm{Hom}(\Lambda^\tau, \mathbb{Z}_2)$. Using the vanishing of the obstruction map $\mathcal{O}$ (see Proposition 3.5) it follows that there exists a τ-Real line bundle $(L', \tilde{\tau}')$ with $c_1(L') = u$. Put $w' := w_1(L'^{\tilde{\tau}'}|_{T^0})$. We know by Remark 3.9 that $w' \in W(u)$. Therefore the difference $w - w'$ vanishes on $(\mathrm{id} + \tau)\Lambda \subset \Lambda^\tau$, so it defines a morphism $v \in \mathrm{Hom}(\Lambda^\tau/(\mathrm{id} + \tau)\Lambda, \mathbb{Z}_2)$. Let $[\chi]$ be

the corresponding element in $[\Lambda^\vee]^{\tau^*}/(\mathrm{id}+\tau^*)\Lambda^\vee$. Then $(L,\tilde\tau):=(L',\tilde\tau')\otimes(L_\chi,\tilde\tau_\chi)$ is a τ-Real line bundle with $cw_0([L,\tilde\tau])=(u,w)$. ∎

4. REAL THETA LINE BUNDLES

4.1. Holomorphic line bundles on a complex torus.
We will see that using the Kobayashi-Hitchin correspondence between abelian Hermite-Einstein connections and holomorphic line bundles one can recover the classical Appell-Humbert theorem in a completely natural way.

Suppose that J is a complex structure on V, endow the torus $T:=V/\Lambda$ with the induced holomorphic structure, and let L be a Hermitian line bundle on T whose Chern class $c_1(L)$ corresponds to $u\in\mathrm{Alt}^2(\Lambda,\mathbb{Z})$. A connection $A\in\mathcal{A}(L)$ is Hermite-Einstein with respect to the flat Kähler metric on T (defined by any Hermitian structure on V) if and only if it is Yang-Mills and its curvature is of type (1,1). Therefore the group $\mathrm{Pic}(T)$ of isomorphism classes of holomorphic line bundles on T can be identified with the union $\coprod_{c_1(L)\in\mathrm{NS}(T)}\mathcal{T}(L)$, where $\mathrm{NS}(T)\subset H^2(T,\mathbb{Z})$ is the Neron-Severi group of T. $\mathrm{NS}(T)$ can be identified with the subgroup of $\mathrm{Alt}^2(\Lambda,\mathbb{Z})$ consisting of forms u whose $\mathbb{R}$-linear extension is J-invariant. This means that the corresponding differentiable form on V is of type (1,1). Our goal is to find a natural holomorphic factor of automorphy ϵ for the holomorphic line bundle $\mathcal{L}(e)$ which corresponds to the Hermite-Einstein connection $A(e)$ on $L(e)$. The holomorphic structure of $\mathcal{L}(e)$ is defined by the semi-connection $\bar\partial_{A(e)}$. The pull-back of this semi-connection to the trivial line bundle $V\times\mathbb{C}$ is given by $\bar\partial_u:=\bar\partial+\theta_u^{0,1}$, so it does not coincide with the trivial semi-connection $\bar\partial$ (unless of course $u=0$). We want to construct explicitly a holomorphic line bundle $\mathcal{L}'(e)$ on T which is holomorphically isomorphic to $\mathcal{L}(e)$ and whose pull-back to V is the standard trivial holomorphic line bundle $(V\times\mathbb{C},\bar\partial)$. This line bundle will be defined by a *holomorphic factor of automorphy* $\epsilon=(\epsilon_\lambda)_{\lambda\in\Lambda}$.

In order to obtain explicitly this holomorphic factor of automorphy, the first step is to find a complex gauge transformation $g\in\mathcal{C}^\infty(V,\mathbb{C}^*)$ such that $g^*(\bar\partial)=\bar\partial_u$, i.e. we have to solve the equation:

$$g^{-1}dg=\theta_u^{0,1}\ .$$

If g is a solution of the this equation, the corresponding factor of automorphy will be

$$e'_\lambda(v):=g(v+\lambda)e_\lambda(v)g^{-1}(v)\ .$$

Using complex coordinates z^j on V and writing u as

$$u=\frac{i}{2}\sum_{j,k}\omega_{jk}dz^j\wedge d\bar z^k$$

with $\bar\omega_{ij}=\omega_{ji}$ one obtains

$$\theta_u^{0,1}=\frac{\pi}{2}\sum_{j,k}\omega_{jk}z^j d\bar z^k\ .$$

Note that $\theta_u^{0,1}=\frac{\pi}{2}\bar\partial(\sum_{j,k}\omega_{jk}z^j\bar z^k)$, so one can take

$$g(v)=e^{\frac{\pi}{2}\sum_{j,k}\omega_{jk}v^j\bar v^k}\ .$$

Recall that the Hermitian form associated with u ($\mathbb{C}$-linear in the second variable) is given by

$$H_u(v, w) := u(v, Jw) + iu(v, w) \ .$$

One checks that $H(v, w) = \sum_{j,k} \omega_{jk} w^j \bar{v}^k$. Therefore the most natural solution is $g(v) = e^{\frac{\pi}{2} H_u(v,v)}$. With this choice of the complex gauge transformation g, the corresponding holomorphic factor of automorphy of $\mathcal{L}'(e) \simeq \mathcal{L}(e)$ is given by:

$$\epsilon_\lambda(v) = g(v + \lambda)g^{-1}(v)e_\lambda(v) = a_\lambda e^{\pi(H(\lambda,v) + \frac{1}{2} H(\lambda,\lambda))} \ .$$

This is the *canonical factor of automorphy* in the sense of Mumford's [Mu]. Note that g defines an isomorphism of Hermitian holomorphic line bundles on V

$$(V \times \mathbb{C}, \bar{\partial}_{A(e)}, h_0) \longrightarrow (V \times \mathbb{C}, \bar{\partial}, |g|^{-2} h_0) \ ,$$

where h_0 is the standard Hermitian metric on the trivial line bundle $V \times \mathbb{C}$. This isomorphism maps the Chern connection on the left (which is $A(e)$) to the Chern connection of the pair $(\bar{\partial}, |g|^{-2} h_0)$, which descends to the Chern connection of $\mathcal{L}'(e)$ endowed with the metric induced by $|g|^{-2} h_0$ (which is the unique Hermite-Einstein connection of the holomorphic line bundle $\mathcal{L}'(e)$). This proves the following important theorem which yields the factor of automorphy e of a Yang-Mills connection which is gauge equivalent to the Hermitian-Einstein connection of a holomorphic line bundle $\mathcal{L}(\epsilon)$ defined by a canonical factor of automorphy ϵ. More precisely

Theorem 4.1. *Let $\epsilon = (\epsilon_\lambda)_{\lambda \in \Lambda}$ with*

$$\epsilon_\lambda(v) = a_\lambda e^{\pi(H(\lambda,v) + \frac{1}{2} H(\lambda,\lambda))}$$

be a canonical factor of automorphy for a holomorphic line bundle $\mathcal{L}(\epsilon)$ on T, where $a : \Lambda \to S^1$ is an S^1-valued $\mathrm{Im}(H)$-character. Then the Yang-Mills connection $A(e)$ defined by the factor of automorphy $e_\lambda(v) = a_\lambda e^{\pi i \mathrm{Im}(H)(\lambda,v)}$ is gauge-equivalent to the unique Hermite-Einstein connection on the holomorphic bundle $\mathcal{L}(\epsilon)$.

In particular $\bar{a}_\lambda \in S^1$ is the holonomy of this Hermite-Einstein connection along the loop $p \circ c_\lambda$ defined by λ.

Note that this theorem allows one to read off the holonomy of the Hermitian-Einstein connection on the holomorphic bundle $\mathcal{L}(\epsilon)$ along segments of the form $p \circ c_\lambda$.

4.2. **Real Riemann theta line bundles.** Put

$$\mathcal{H}_g := \{Z \in M_g(\mathbb{C}) | \ Z = Z^t, \ \mathrm{Im}(Z) > 0\} \ .$$

To an element $Z \in \mathcal{H}_g$ one can associate

  (1) the lattice

$$\Lambda_Z := \langle Z, I_g \rangle \subset \mathbb{C}^g \ ,$$

    which is generated by the columns of Z and the elements e_i of the standard basis of $\mathbb{C}^g$,
  (2) the complex torus $X_Z := \mathbb{C}^g / \Lambda_Z$,
  (3) the Hermitian form $H_Z : \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C}$ defined by

$$H_Z(v, w) := \bar{v}^t \mathrm{Im}(Z)^{-1} w \ .$$

The assignment

$$Z \mapsto (X_Z, H_Z, \text{ columns of } (Z, I_g))$$

defines an identification between the Siegel upper half space $\mathcal{H}_g$ and the moduli space of principally polarized abelian varieties X endowed with a symplectic basis of $H_1(X, \mathbb{Z})$ (see [5], Prop. 8.1.2).

Any element $\lambda \in \Lambda_Z$ decomposes as

$$\lambda = Za + b \ , \ a, \ b \in \mathbb{Z}^g \ .$$

Note that the the restriction $E_Z := \text{Im}(H_Z)\big|_{\Lambda_Z \times \Lambda_Z}$ of imaginary part of H_Z to Λ_Z is given by

$$(12) \qquad\qquad E_Z(\lambda, \lambda') = b^t a' - a^t b' \in \mathbb{Z} \ ,$$

so it is an element of $\text{Alt}^2(\Lambda_Z, \mathbb{Z})$. The summands $\langle Z \rangle$, $\langle I_g \rangle$ are isotropic for E_Z, and the matrix of E_Z in the basis $(Z_1, \ldots, Z_g, e_1, \ldots, e_g)$ is the standard skew-symmetric symplectic matrix

$$S_g := \begin{bmatrix} 0_g & -I_g \\ I_g & 0_g \end{bmatrix} \ .$$

The classical Riemann theta function associated with Z is the holomorphic function $\theta_Z : \mathbb{C}^g \to \mathbb{C}$ given by

$$\theta_Z(z) := \sum_{l \in \mathbb{Z}^g} e^{2\pi i(\frac{1}{2} l^t Z l + l^t z)} \ .$$

This function satisfies the functional equations

$$\theta_Z(z + Za + b) = e^{-2\pi i(\frac{1}{2} a^t Za + a^t Z)} \theta_Z(z) \ , \ a, \ b \in \mathbb{Z}^g \ ,$$

which show that it descends to a holomorphic section in the line bundle $\mathcal{L}_Z$ on X_Z defined by the *classical factor of automorphy*

$$e_Z : \Lambda_Z \times \mathbb{C}^g \longrightarrow \mathbb{C}$$

$$(Za + b, z) \mapsto e^{-2\pi i(\frac{1}{2} a^t Za + a^t z)} \ .$$

The zero locus of this section defines the Riemann theta divisor $\Theta_Z \subset X_Z$. Note that Θ_Z is symmetric, i.e.

$$(-1)^* \Theta_Z = \Theta_Z \ .$$

The first Chern class $c_1(\mathcal{L}_Z)$ of the theta line bundle $\mathcal{L}_Z$ is

$$c_1(\mathcal{L}_Z) = E_Z \in \text{Alt}^2(\Lambda_Z, \mathbb{Z}) = H^2(X_Z, \mathbb{Z}) \ .$$

Note that the map $\chi_Z : \Lambda_Z \to \{\pm 1\}$ defined by

$$\chi_Z(Za + b) := e^{\pi i a^t b}$$

is an E_Z-character. In fact, χ_Z is the standard E_Z-character defined by the E_Z-decomposition $\Lambda_Z = \langle Z \rangle \oplus \langle I_g \rangle$, i.e. it holds

$$\chi_Z(Za + b) = e^{\pi i E_Z(Za, b)}$$

(see [5], section 3.1).

Proposition 4.2. *The Appel-Humbert datum defining the line bundle $\mathcal{L}_Z$ is the pair (H_Z, χ_Z).*

Proof: Consider the holomorphic function

$$b_Z(z) := e^{\frac{\pi}{2} z^t \mathrm{Im}(Z)^{-1} z} \ ,$$

and the new factor of automorphy e'_Z defined by

$$e'_Z(\lambda, z) := e_Z(\lambda, z) b_Z(z + \lambda) b_Z(z)^{-1} \ .$$

One checks that

$$e'_Z(\lambda, z) = \chi_Z(\lambda) e^{\pi(\frac{1}{2} H_Z(\lambda, \lambda) + H_Z(\lambda, z))} \ .$$

■

We recall that a *Real principally polarized abelian variety* is a principally polarized abelian variety (X, E) endowed with a real structure $\tau : X \to X$ such that $\tau^*(E) = -E$.

Using Theorem 4.1 of [19] one sees that any Real principally polarized abelian variety can be represented by a triple

$$(X_Z, H_Z, \ \text{columns of } (Z, I_g)) \ ,$$

where $Z = \frac{1}{2} M + iS$, S is a symmetric, positive definite real matrix and M has one of the following forms:

(1)

$$M = \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} \ , \ s > 0 \ ,$$

(2)

$$M = \begin{pmatrix} J_s & 0 \\ 0 & 0 \end{pmatrix} \text{ with } s \text{ even where } J_s := \begin{bmatrix} 0 & & & & 1 \\ & & & \cdot^{\cdot^{\cdot}} & \\ & & 1 & & \\ & \cdot^{\cdot^{\cdot}} & & & \\ 1 & & & & 0 \end{bmatrix}$$

The first type is called diasymmetric, whereas the second type is called orthosymmetric. The Real structure τ on X is induced by standard conjugation $^-$ on $\mathbb{C}^g$. It is easy to check that the form $E_Z \in \mathrm{Alt}^2(\Lambda, \mathbb{Z})$ defined by formula (12) is indeed Real with respect to this standard Real structure. Note however that the line bundle $\mathcal{L}_Z$ associated with Z is *not always* Real. It is not Real when M has the first form, but it is Real when M has the second form. In the first case the pull-back of $\mathcal{L}_Z$ via the translation by $\left[-\frac{1}{2} \sum_{j=1}^s Z_i \right]$ becomes Real. Indeed, the transformation rule of the Appel-Humbert datum with respect to translations is given by [5] Lemma 2.3.3. Taking into account this formula, the condition that $t^*_{[v]}(\mathcal{L}_Z)$ is Real becomes:

$$\chi_Z(\tau \lambda) = \overline{\chi_Z(\lambda)} e^{-2\pi i E_Z(v, \lambda + \tau(\lambda))} \ \forall \lambda \in \Lambda_Z \ ,$$

which can be easily checked. Note that $\Lambda_Z^- = \{ Za + b \in \Lambda_Z | \ a = 0 \}$.

Theorem 4.3. *Let $\mathcal{L}_Z$ be the Riemann theta line bundle associated with a matrix $Z = \frac{1}{2} M + S$, where M has one of the tow standard forms above.*

(1) *When Z is diasymmetric then $t^*_{[v]}(\mathcal{L}_Z)$ is Real for $v := -\frac{1}{2} \sum_{j=1}^s Z_i$, and*

$$cw_0([t^*_{[v]}(\mathcal{L}_Z)]) = (E_Z, p_s) \ ,$$

where $p_s : \Lambda_Z^- \to \mathbb{Z}_2$ is given by $p_s(b) := \sum_{i=1}^s b_i \ (mod \ 2)$.

(2) *When Z is orthosymmetric, then $\mathcal{L}_Z$ has a natural Real structure, and*

$$cw_0([\mathcal{L}_Z]) = (E_Z, 0) .$$

Proof. Using Lemma 2.3.3 in [5] one sees that the Appel-Humbert data describing $t^*_{[v]}(\mathcal{L}_Z)$ are $(H_Z, \chi_{Z,v})$ with

$$\chi_{Z,v}(Za + b) = e^{\pi i(a^t b + \sum_{j=1}^s b_j)} .$$

The result in both cases follows now from Proposition 3.10. ∎

Note that using the difference formula given by Proposition 3.8 one obtains an explicit formula for the first Stiefel-Whitney classes of the restrictions of the corresponding real line bundles to every component of X_Z^τ.

4.3. **Theta line bundles of Klein surfaces.** Let (C, ι) be a Klein surface of genus g, and $\Theta \subset \mathrm{Pic}^{g-1}(C)$ the geometric theta divisor defined by

$$\Theta := \{\mathcal{L} \in \mathrm{Pic}^{g-1}(C)|\ h^0(\mathcal{L}) > 0\} .$$

Denote by

$$S(C) := \{\kappa \in \mathrm{Pic}^{g-1}(C)|\ \kappa^{\otimes 2} = \omega_C\}$$

the set of theta characteristics of C. This set is naturally a $\mathrm{Pic}^0(C)_2$-torsor. For any $\kappa \in S(C)$ we consider the divisor

$$\Theta_\kappa := \Theta - \kappa \subset \mathrm{Pic}^0(C)$$

which will be called the theta divisor associated with κ. Note that

$$(-1)^*\Theta_\kappa = \Theta_\kappa .$$

Denote by $\hat{\iota}$ the involution $\hat{\iota} : \mathrm{Pic}(C) \to \mathrm{Pic}(C)$ given by

$$\hat{\iota}(\mathcal{L}) = \overline{\iota^*(\mathcal{L})} .$$

Note that $\hat{\iota}(\mathrm{Pic}^d(C)) = \mathrm{Pic}^d(C)$ for any $d \in \mathbb{Z}$ and that $\hat{\iota}$ leaves invariant the geometric theta divisor Θ (because the $H^0(C, \mathcal{L})$ and $H^0(C, \hat{\iota}(\mathcal{L}))$ are naturally anti-isomorphic).

Clearly if a theta characteristic $\kappa \in S(C)$ is ι-Real (i.e. $\hat{\iota}(\kappa) = \kappa$) then one has

$$\hat{\iota}(\Theta_\kappa) = \Theta_\kappa .$$

The set of $\hat{\iota}$-Real theta characteristics is non-empty; this set has been studied in [9]. We obtain a holomorphic line bundle

$$\mathcal{L}_\kappa := \mathcal{O}_{\mathrm{Pic}^0(C)}(\Theta_\kappa)$$

on $\mathrm{Pic}^0(C)$, which is symmetric in the sense that

$$(-1)^*\mathcal{L}_\kappa = \mathcal{L}_\kappa .$$

Note also that $\mathcal{L}_\kappa$ is naturally a $\hat{\iota}$-Real line bundle on $\mathrm{Pic}^0(C)$ since it is associated with a Real divisor. The first Chern class of $\mathcal{L}_\kappa$ is the element

$$u_C \in H^2(\mathrm{Pic}^0(C), \mathbb{Z}) = \mathrm{Alt}^2(H^1(C, \mathbb{Z}), \mathbb{Z})$$

defined by the cup form $u_C : H^1(C, \mathbb{Z}) \times H^1(C, \mathbb{Z}) \to \mathbb{Z}$.

Our newt goals are:

(1) determine explicitly the Appel-Humbert data of $\mathcal{L}_\kappa$,
(2) determine explicitly the element of $H^1_{\mathbb{Z}_2}(\mathrm{Pic}^0(C), \underline{S}^1(1))$ defined by the $\hat{\iota}$-Real line bundle $\mathcal{L}_\kappa$ on $\mathrm{Pic}^0(C)$

Clearly the first component of the Appel-Humbert datum defining $\mathcal{L}_\kappa$ is the Hermitian form H_{u_C} associated with the cup form u_C. The second component χ_κ of the datum is an u_C-character which takes values in $\{\pm 1\}$ since $\mathcal{L}_\kappa$ is symmetric (see Corollary 2.3.7 in [5]). We obtain the identities

$$\chi_\kappa(\lambda + \lambda') = \chi_\kappa(\lambda)\chi_\kappa(\lambda')e^{\pi i u_C(\lambda,\lambda')} \ .$$

Note that χ_κ is trivial on $2H^1(C,\mathbb{Z}) \subset H^1(C,\mathbb{Z})$ and descends to a well-defined map

$$\bar{\chi}_\kappa : {H^1(C,\mathbb{Z})}\big/{2H^1(C,\mathbb{Z})} = H^1(C,\mathbb{Z}_2) \longrightarrow \{\pm 1\} = \mathbb{Z}_2 \ .$$

This $\mathbb{Z}_2$-valued map satisfies the identity

$$\bar{\chi}_\kappa([\lambda] + [\lambda']) = \bar{\chi}_\kappa([\lambda]) + \bar{\chi}_\kappa([\lambda']) + \overline{u_C(\lambda,\lambda')} \ ,$$

Hence $\overline{\chi}_\kappa$ is a quadratic refinement of the (mod 2) cup form

$$\bar{u}_C : H^1(C,\mathbb{Z}_2) \times H^1(C,\mathbb{Z}_2) \to \mathbb{Z}_2 \ .$$

Let $\kappa \in S(C)$. Mumford defines a map $q_\kappa : \text{Pic}^0(C)_2 \to \mathbb{Z}_2$ given by

$$\eta \mapsto h^0(\kappa \otimes \eta) - h^0(\kappa) \ (\text{mod } 2) \ .$$

With the canonical identification $\text{Pic}^0(C)_2 = H^1(C,\mathbb{Z}_2)$ and using Poincaré duality Mumford's theta form q_κ becomes a map (denoted by the same symbol)

$$q_\kappa : H_1(C,\mathbb{Z}_2) \to \mathbb{Z}_2$$

which satisfies *the Riemann-Mumford relations:*

(13) $$q_\kappa(\eta + \eta') = q_\kappa(\eta) + q_\kappa(\eta') + \eta \cdot \eta'$$

Proposition 4.4. *Suppose $\mathcal{L}_\kappa$ is associated with the Appel-Humbert data (H_{u_C}, χ_κ). Then*

$$\chi_\kappa(\lambda) = (-1)^{q_\kappa(\overline{\lambda \cap [C]})} \ \forall \lambda \in H^1(C,\mathbb{Z}) \ .$$

Proof. Using [5] Proposition 4.7.2. one obtains

$$\chi_\kappa(\lambda) = (-1)^{\text{mult}_{[\frac{1}{2}\lambda]}(\Theta_\kappa) - \text{mult}_{[0]}(\Theta_\kappa)} \ \forall \lambda \in H^1(C,\mathbb{Z}) \ .$$

Now we use the Riemann singularity theorem [5]:

$$\text{mult}_{[\mathcal{L}]}(\Theta) = h^0(\mathcal{L}) \ .$$

One obviously has

$$\text{mult}_{[0]}(\Theta_\kappa)) = \text{mult}_{[\kappa]}(\Theta)) = h^0(\kappa)$$

$$\text{mult}_{[\frac{1}{2}\lambda]}(\Theta_\kappa) = \text{mult}_{\kappa \otimes [\frac{1}{2}\lambda]}(\Theta)) = h^0\left(\kappa \otimes \left[\frac{1}{2}\lambda\right]\right) \ .$$

Therefore

$$\chi_\kappa(\lambda) = (-1)^{q_\kappa([\frac{1}{2}\lambda])} \ .$$

Since the image of $\left[\frac{1}{2}\lambda\right] \in \text{Pic}^0(C)_2$ in $H_1(C,\mathbb{Z}_2)$ via our identification is $\overline{\lambda \cap [C]})$ we get

$$\chi_\kappa(\lambda) = (-1)^{q_\kappa(\overline{\lambda \cap [C]})} \ .$$

$\blacksquare$

Composing the map $\mathfrak{F}_X$ defined in Proposition 2.12 with the morphism

$$H^1_{\mathbb{Z}_2}(X, \underline{S}^1(1)) \to H^1(X^\iota, \mathbb{Z}_2)$$

which maps a ι-Real line bundle $(L, \tilde{\iota})$ to the first Stiefel-Whitney class $w_1(L^{\tilde{\iota}})$ one obtains a morphism

$$w : \mathrm{Pic}(X)^{\hat{\iota}} \to H^1(X^\iota, \mathbb{Z}_2) .$$

Let $C_1, \ldots, C_r$ be the connected components of the fixed point locus.
We shall prove the following

Theorem 4.5. *Let $C_1, \ldots, C_r$ be the connected components of the fixed point locus C^ι. One has*

$$q_\kappa([C_i]_2) = \langle w(\kappa), [C_i] \rangle + 1 .$$

In order to prove this theorem we need some preparations:

Let (C, g) be a closed, connected oriented Riemann surface. We identify the $SO(2)$-frame bundle $P_g \to C$ of (C, g) with the sphere bundle $q : S(T_C) \to C$ in the natural way, and we fix a Spin-structure $\sigma : Q \to S(T_C)$ of C. A simple oriented closed curve c in C yields a simple closed curve $\tilde{c} \subset S(T_C)$ given by the unit tangent vectors of c which are compatible with the orientation. This defines a homology class $[\tilde{c}] \in H_1(S(T_C), \mathbb{Z}_2)$. Changing the orientation of c will give a different lift in $S(T_C)$, but the $\mathbb{Z}_2$-homology classes defined by the two lifts coincide. Any homology class $\eta \in H_1(C, \mathbb{Z}_2)$ can be represented by a union of pairwise disjoint simple closed curves $c_1, \ldots, c_m$. We will need the following important *lifting result* of Johnson [11]:

Let $z \in H_1(S(T_C), \mathbb{Z}_2)$ be the homology class of a fibre. Putting

$$\hat{\eta} := \sum_{i=1}^m [\tilde{c}_i] + mz$$

defines a canonical lifting map $\hat{} : H_1(C, \mathbb{Z}_2) \to H_1(S(T_C), \mathbb{Z}_2)$ satisfying the identity

$$\widehat{\eta + \eta'} = \hat{\eta} + \hat{\eta}' + (\eta \cdot \eta')z .$$

Now let $\xi \in H^1(S(T_C), \mathbb{Z}_2)$ be the class of a Spin-structure on C. This is equivalent to the condition $\langle \xi, z \rangle = 1$. Johnson defines a map $\omega_\xi : H_1(C, \mathbb{Z}_2) \to \mathbb{Z}_2$ by

$$\omega_\xi(\eta) := \langle \xi, \hat{\eta} \rangle ,$$

where the canonical lift $\hat{\eta}$ is defined by a Spin-structure $\sigma : Q \to S(T_C)$ in the class ξ. This map satisfies the identity

$$\omega_\xi(\eta + \eta') = \omega_\xi(\eta) + \omega_\xi(\eta') + \eta \cdot \eta' .$$

In other words ω_ξ is a quadratic refinement of the (mod 2) intersection form.

Let $Q(H_1(C, \mathbb{Z}_2), \cdot)$ be the set of quadratic refinements of the (mod 2) intersection form and denote by $\mathrm{Spin}(C)$ the set of equivalence classes of Spin-structures on C. Note that there is a well known bijection $\xi : S(C) \to \mathrm{Spin}(C)$ between the set of theta characteristics and the set of equivalence classes of Spin-structures on C. We have just explained that Johnson's construction defines a map

$\omega : \mathrm{Spin}(C) \to Q(H_1(C, \mathbb{Z}_2), \cdot)$. On the other hand, by the Riemann-Mumford relations, Mumford's construction defines a map $q : S(C) \to Q(H_1(C, \mathbb{Z}_2), \cdot)$. Libgober has shown [13] that the following diagram commutes:

$$
\begin{array}{ccc}
S(C) & & \\
& \searrow^{q} & \\
\xi \downarrow & & Q(H_1(C, \mathbb{Z}_2), \cdot) \\
& \nearrow^{\omega} & \\
\mathrm{Spin}(C) & &
\end{array}
$$

More precisely one has

(14) $$q_\kappa(\eta) = \omega_{\xi_\kappa}(\eta) \ \forall \eta \in H_1(C, \mathbb{Z}_2) \ .$$

Now let g be a ι-invariant Hermitian metric on C and denote again by $S(T_C)$ the sphere bundle of the real tangent bundle T_C of C. Let κ be a holomorphic line bundle representing a theta characteristic. We choose a holomorphic isomorphism $\phi : \kappa^{\otimes 2} \to \mathcal{K}_C$ with the canonical line bundle $\mathcal{K}_C$, and we endow κ with the Hermitian metric induced via ϕ from the real cotangent bundle T_C^*, which is the underlying differentiable line bundle of $\mathcal{K}_C$. Via the standard identification $P_g = S(T_C)$ the Spin-structure associated with κ is the double cover

$$\sigma : S(\kappa) \xrightarrow{\otimes 2} S(\kappa^{\otimes 2}) \xrightarrow{\phi} S(T_C^*) \xrightarrow{\simeq} S(T_C) \ .$$

Note that, by the definition of the map ξ, σ represents the class $\xi_{[\kappa]}$. The sphere bundle $S(\kappa)$ can be also regarded as an S^1-bundle over C via the composition $\rho := q \circ \sigma$. Note that the holomorphic line bundle $\mathcal{K}_C$ comes with a *canonical* anti-holomorphic ι-Real structure $\tilde{\iota}_{\mathrm{can}}$ acting on local holomorphic 1-forms by $\eta \mapsto \iota^*(\bar{\eta})$. The induced involution on $S(T_C)$ is just the tangent map ι_* of ι.

Remark 4.6. *There exists an anti-holomorphic ι-Real structure $\tilde{\iota}_0$ on κ which lifts the canonical ι-Real structure $\tilde{\iota}_{\mathrm{can}}$ on $\mathcal{K}_C$ via $\phi \circ (\cdot)^{\otimes 2}$. This ι-Real structure is unique up to sign.*

Indeed, Let $\tilde{\iota}$ be any anti-holomorphic ι-Real structure on κ (see Proposition 2.12). The ι-Real structure induced on $\mathcal{K}_C$ via $\phi \circ (\cdot)^{\otimes 2}$ is well-defined and anti-holomorphic, so it is equivalent with $\tilde{\iota}_{\mathrm{can}}$ modulo a constant $\zeta \in S^1$. It suffices to put $\tilde{\iota}_0 := z\tilde{\iota}$, where z is a square root of ζ. ∎

Lemma 4.7. *Let κ be a theta characteristic endowed with a ι-Real structure $\tilde{\iota}_0$ lifting $\tilde{\iota}_{\mathrm{can}}$, and let $\sigma : S(\kappa) \to S(T_C)$ be the associated Spin-structure. Let $\gamma : S^1 \to C^\iota$ be a parametrization with unit speed of a connected component $C_0 \subset C^\iota$, and let $\Gamma_0 \subset S(T_C)$ be the image of the tangent map γ_*. Replacing $\tilde{\iota}_0$ by $-\tilde{\iota}_0$ if necessary the following holds:*

(1) $\sigma(S(\kappa)^{\tilde{\iota}_0}\big|_{C_0}) = \Gamma_0$ and the obvious restrictions of σ, ρ and q define a commutative diagram

$$
\begin{array}{ccc}
S(\kappa)^{\tilde{\iota}_0}\big|_{C_0} & \xRightarrow{=} & S(\kappa)^{\tilde{\iota}_0}\big|_{C_0} \\
\downarrow \sigma_0 & & \downarrow \rho_0 \\
\Gamma_0 & \xrightarrow{q|_{\Gamma_0}} & C_0
\end{array}
$$

(2) *The principal $\mathbb{Z}_2$-bundles $\sigma_0 : S(\kappa)^{\tilde{\iota}_0}\big|_{C_0} \to \Gamma_0$, $\rho_0 : S(\kappa)^{\tilde{\iota}_0}\big|_{C_0} \to C_0$ are isomorphic via $q_{|\Gamma_0}$.*

Proof. Note first that the restriction $S(T_C)^{\iota_*}\big|_{C_0}$ of the fixed point locus $S(T_C)^{\iota_*}$ to C_0 is the disjoint union $\Gamma_0 \cup -\Gamma_0$.

Claim: Either $\sigma^{-1}(\Gamma_0) = S(\kappa)^{\tilde{\iota}_0}\big|_{C_0}$, or $\sigma^{-1}(-\Gamma_0) = S(\kappa)^{\tilde{\iota}_0}\big|_{C_0}$.

Indeed, for any $l \in S(\kappa)^{\tilde{\iota}_0}\big|_{C_0}$ one has $\sigma(l) \in S(T_C)^{\iota_*}\big|_{C_0}$, because $\tilde{\iota}_0$ is a lift of ι_* via σ. Therefore either $\sigma(l) \in \Gamma_0$, or $\sigma(l) \in -\Gamma_0$. Two elements l_1, $l_2 \in S(\kappa)^{\tilde{\iota}_0}\big|_{C_0}$ have the same image via σ if and only if $l_2 = \pm l_1$ (because σ is equivariant with respect to $S^1 \xrightarrow{(\cdot)^2} S^1$ and is fibrewise equivalent to the standard double cover of S^1). Therefore the quotient $S(\kappa)^{\tilde{\iota}_0}\big|_{C_0}/\{\pm\mathrm{id}\}$ is mapped injectively to $S(T_C)^{\iota_*}\big|_{C_0}$. On the other hand the three projections

$$S(\kappa)^{\tilde{\iota}_0}\big|_{C_0}/\{\pm\mathrm{id}\} \to C_0 \ , \ \Gamma_0 \to C_0 \ , \ -\Gamma_0 \to C_0$$

are all diffeomorphisms, so the image of $S(\kappa)^{\tilde{\iota}_0}\big|_{C_0}/\{\pm\mathrm{id}\}$ via σ must be a section of $S(T_C)^{\iota_*}\big|_{C_0} \to C_0$, which proves the claim.

If we replace $\tilde{\iota}_0$ by $-\tilde{\iota}_0$ the new fixed point locus $S(\kappa)^{-\tilde{\iota}_0}$ will be $iS(\kappa)^{\tilde{\iota}_0}$, and the multiplication by i on $S(\kappa)$ corresponds to the multiplication by -1 on $S(T_C)$. Therefore, replacing $\tilde{\iota}_0$ by $-\tilde{\iota}_0$ if necessary, we can assume that $\sigma^{-1}(\Gamma_0) = S(\kappa)^{\tilde{\iota}_0}\big|_{C_0}$. This proves the first statement. The second follows directly from the first. $\blacksquare$

Corollary 4.8. *Let $\xi_\kappa \in H^1(S(T_C), \mathbb{Z}_2)$ be the first Stiefel-Whitney class of the $\mathbb{Z}_2$-bundle $\sigma : S(\kappa) \to S(T_C)$. Then, for every connected component C_0 of C^ι one has*

$$\langle \xi_\kappa, [\Gamma_0] \rangle = \langle w(\kappa), [C_0] \rangle \ .$$

Now the proof of Theorem 4.5 is immediate:

Proof. (of Theorem 4.5) Using Libgober's formula (14) and Corollary 4.8 one obtains:

$$q_\kappa([C_i]_2) = \omega_{\xi_\kappa}([C_i]_2) = \langle \xi_\kappa, \widetilde{[C_i]}_2 + z \rangle = \langle \xi_\kappa, \widetilde{[C_i]}_2 \rangle + 1 = \langle w(\kappa), [C_i]_2 \rangle + 1 \ .$$

$\blacksquare$

Corollary 4.9. *One has $q_\kappa([C^\iota]_2) \equiv s \pmod{2}$, where s stands for the Comessatti characteristic of (C, ι).*

Proof. The Comessatti characteristic of (C, ι) is given by the formula $s = g + 1 - r$ (see [19]). On the other hand, using the results in Appendix B it follows:

$$\langle w(\kappa), [C^\iota]_2 \rangle = \langle c_1(\kappa), [C] \rangle \pmod{2}$$

Applying Theorem 4.5 we get:

$$q_\kappa([C^\iota]_2) = \langle w(\kappa), [C^\iota]_2 \rangle + r \ (\bmod\ 2) \equiv g - 1 + r \ (\bmod\ 2) \equiv s \ (\bmod\ 2) \ .$$

$\blacksquare$

Recall [9] that the orientation obstruction $a(C, \iota)$ is 0 when $C/\langle\iota\rangle$ is orientable and $a(C, \iota)$ is 1 when not. One has $a(C, \iota) = 0$ if and only if $C \setminus C^\iota$ has two connected components.

Note that the submodule $\langle\{[C_i]| \ 1 \le i \le r\}\rangle_{\mathbb{Z}}$ generated by the classes of the circles C_i is obviously contained in the ι_*-invariant submodule $H_1(C, \mathbb{Z})^{\iota_*}$ of $H_1(C, \mathbb{Z})$. We will see that this submodule together with the submodule $(\mathrm{id} + \iota_*)H_1(C, \mathbb{Z})$ of trivial invariants generates $H_1(C, \mathbb{Z})^{\iota_*}$. We refer to [15] for the following technical result:

Lemma 4.10. *Let (C, ι) be a Klein surface of genus g, r the number of connected components of C^ι, and denote by $s := g - r + 1$ its Comessatti characteristic. Choose an orientation of C^ι, introduce an order relation $(C_1, \ldots, C_r)$ on its set of connected components, and put $v_i := [C_i] \in H_1(C, \mathbb{Z})$.*

(1) *Suppose that the quotient $C/\langle\iota\rangle$ is orientable, and put $k := \frac{s}{2}$*

    (a) *The homology group $H_1(C, \mathbb{Z})$ admits a symplectic basis of the form*

$$(v_1, \ldots, v_{r-1}, x_1, \ldots, x_k, \iota_* x_1, \ldots, \iota_* x_k, w_1, \ldots w_{r-1}, y_1, \ldots, y_k, -\iota_* y_1, \ldots, -\iota_* y_k)$$

        where $\iota_* w_i = -w_i$.

    (b) *The associated basis*

$$(\underline{a}, \underline{b}) := (v_1, \ldots, v_{r-1}, x_1 + \iota_* x_1, \ldots, x_k + \iota_* x_k, y_1 + \iota_* y_1, \ldots, y_k + \iota_* y_k,$$

$$, w_1, \ldots, w_{r-1}, y_1, \ldots, y_k, \iota_* x_1, \ldots, \iota_* x_k)$$

        is symplectic, its first g basis vectors $(a_1, \ldots, a_g)$ are ι_*-invariant, whereas the last g basis vectors $(b_1, \ldots, b_g)$ satisfy:

$$\iota_* b_i = -b_i \text{ for } 1 \le i \le r - 1, \ \iota_* b_j = a_{k+j} - b_j \text{ for } r \le j \le r + k - 1,$$

$$\iota_* b_l = a_{l-k} - b_l \text{ for } k + r \le l \le 2k + r - 1 = g$$

(2) *Suppose that the quotient $C/\langle\iota\rangle$ is not orientable. There exists a symplectic basis*

$$(\underline{a}, \underline{b}) = (a_1, \ldots, a_g, b_1, \ldots, b_g)$$

    of $H_1(C, \mathbb{Z})$ such that $a_i = v_i$ for $1 \le i \le r$, the first g basis vectors $(a_1, \ldots, a_g)$ are ι_*-invariant, whereas the last g basis vectors $(b_1, \ldots, b_g)$ satisfy the identities:

$$\iota_* b_j = \begin{cases} -b_j - \sum_{i=1}^g a_i & \text{for} \quad 1 \le j \le r \\[2mm] -b_j - a_j - \sum_{i=1}^g a_i & \text{for} \quad r + 1 \le j \le g \end{cases}$$

Lemma 4.11. *Let (C, ι) be a Klein surface with $C^\iota \ne \emptyset$.*

(1) *The natural map*

$$j : \langle\{v_i| \ 1 \le i \le r\}\rangle_{\mathbb{Z}} \oplus \left[(\mathrm{id} + \iota_*)H_1(C, \mathbb{Z})\right] \longrightarrow H_1(C, \mathbb{Z})^{\iota_*}$$

    is always surjective.

(2) *If $a(C, \iota) = 0$ then, orienting the curves C_i in a suitable way, one has a short exact sequence*

$$1 \longrightarrow \mathbb{Z}\sum_{i=1}^r v_i \longrightarrow \oplus_{i=1}^r \mathbb{Z}v_i \longrightarrow \langle v_1, \ldots, v_r\rangle_{\mathbb{Z}} \to 0 \ ,$$

    and

$$\langle v_1, \ldots, v_r\rangle_{\mathbb{Z}} \cap \left[(\mathrm{id} + \iota_*)H_1(C, \mathbb{Z})\right] = \langle 2v_1, \ldots, 2v_r\rangle_{\mathbb{Z}} \ .$$

(3) *If $a(C, \iota) = 1$, then the canonical epimorphism*

$$\oplus_{i=1}^{r} \mathbb{Z} v_i \longrightarrow \langle v_1, \dots, v_r \rangle_{\mathbb{Z}}$$

is an isomorphism and one has:

$$H_1(C, \mathbb{Z})^{\iota_*} = \langle a_1, \dots, a_g \rangle, \quad (\mathrm{id} + \iota_*) H_1(C, \mathbb{Z}) = \langle 2a_1, \dots, 2a_g, a_{r+1}, \dots, a_g, \sum_{i=1}^{g} a_i \rangle_{\mathbb{Z}}$$

$$= \langle 2a_1, \dots, 2a_g, a_{r+1}, \dots, a_g, \sum_{i=1}^{r} a_i \rangle_{\mathbb{Z}},$$

$$(\mathrm{id} + \iota_*) H_1(C, \mathbb{Z}) \cap \langle a_1, \dots, a_r \rangle = \langle 2a_1, \dots, 2a_r, \sum_{i=1}^{r} a_i \rangle_{\mathbb{Z}} = \langle 2a_1, \dots, 2a_{r-1}, \sum_{i=1}^{r} a_i \rangle_{\mathbb{Z}}.$$

Moreover, writing the sum $\sum_{i=1}^{r} v_i$ in the form $\sum_{i=1}^{r} v_i = x + \iota_*(x)$ (as a trivial invariant), one has $x \cdot \iota_*(x) \equiv s \pmod{2}$, where s is the Comessatti characteristic of the pair $(H_1(C, \mathbb{Z}), \iota_*)$.

Proof. Suppose first that $a(C, \iota) = 0$. In this case the union of circles $C^{\iota} = \cup_{i=1}^{r} C_i$ is the boundary of the closure of one of the connected components of $C \setminus C^{\iota}$ so, orienting these circles as boundary components, one has $\sum_{i=1}^{r} [C_i] = 0$ in $H_1(C, \mathbb{Z})$. We can choose a symplectic basis as in Lemma 4.10 above. It is easy to check that $H_1(C, \mathbb{Z})^{\iota_*}$ decomposes as

$$H_1(C, \mathbb{Z})^{\iota_*} = \langle v_1, \dots, v_{r-1} \rangle_{\mathbb{Z}} \oplus \langle x_1 + \iota_* x_1, \dots, x_k + \iota_* x_k, y_1 + \iota_* y_1, \dots, y_k + \iota_* y_k \rangle_{\mathbb{Z}},$$

whereas the submodule of trivial invariants $(\mathrm{id} + \iota_*) H_1(C, \mathbb{Z})$ decomposes as

$$(\mathrm{id} + \iota_*) H_1(C, \mathbb{Z}) = \langle 2v_1, \dots, 2v_{r-1} \rangle_{\mathbb{Z}} \oplus \langle x_1 + \iota_* x_1, \dots, x_k + \iota_* x_k, y_1 + \iota_* y_1, \dots, y_k + \iota_* y_k \rangle_{\mathbb{Z}}$$

$$= \langle 2v_1, \dots, 2v_r \rangle_{\mathbb{Z}} \oplus \langle x_1 + \iota_* x_1, \dots, x_k + \iota_* x_k, y_1 + \iota_* y_1, \dots, y_k + \iota_* y_k \rangle_{\mathbb{Z}}.$$

Suppose now that $a(C, \iota) = 1$. We make again use of Lemma 4.10. In this case all classes $v_i = [C_i]$ are basis vectors, so they are linearly independent in $H_1(C, \mathbb{Z})$, which proves the first claim. It is easy to see that $H_1(C, \mathbb{Z})^{\iota_*} = \langle a_1, \dots, a_g \rangle$ (see [15]). In this case we obtain

$$(\mathrm{id} + \iota_*) H_1(C, \mathbb{Z}) = \langle (\mathrm{id} + \iota_*)(a_1), \dots, (\mathrm{id} + \iota_*)(a_g), (\mathrm{id} + \iota_*)(b_1), \dots, (\mathrm{id} + \iota_*)(b_g) \rangle$$

$$= \left\langle 2a_1, \dots, 2a_g, \sum_{i=1}^{g} a_i, a_{r+1} + \sum_{i=1}^{g} a_i, \dots, a_g + \sum_{i=1}^{g} a_i \right\rangle$$

$$= \left\langle 2a_1, \dots, 2a_g, \sum_{i=1}^{g} a_i, a_{r+1}, \dots, a_g \right\rangle = \left\langle 2a_1, \dots, 2a_g, \sum_{i=1}^{r} a_i, a_{r+1}, \dots, a_g \right\rangle,$$

which proves the other claims concerning the subgroup $(\mathrm{id} + \iota_*) H_1(C, \mathbb{Z})$ of trivial invariants.

Finally, one has

$$\sum_{i=1}^{r} v_i = \sum_{i=1}^{r} a_i = \sum_{i=1}^{g} a_i - \sum_{j=r+1}^{g} a_j = \sum_{i=1}^{g} a_i + (g - r) \sum_{i=1}^{g} a_i + \sum_{j=r+1}^{g} (b_j + \iota_* b_j)$$

$$= -(g - r + 1)(b_1 + \iota_* b_1) + \sum_{j=r+1}^{g} (b_j + \iota_* b_j),$$

so, we can choose $x = -(g - r + 1)b_1 + \sum_{j=r+1}^{g} b_j$, and

$$x \cdot x \equiv (g - r + 1)b_1 \cdot (\iota_* b_1) + \sum_{j=r+1}^{g} b_j \cdot (\iota_* b_j) \equiv g - r + 1 \pmod 2 \ .$$

∎

The torus $\mathrm{Pic}^0(C)$ can be identified with the quotient $H^1(C, \mathcal{O})/H^1(C, \mathbb{Z})$, where $H^1(X, \mathbb{Z})$ is embedded in $H^1(X, \mathcal{O})$ via the composition

$$H^1(C, \mathbb{Z}) \xrightarrow{\cong} 2\pi i H^1(C, \mathbb{Z}) \hookrightarrow iH^1(C, \mathbb{R}) \hookrightarrow H^1(X, \mathbb{C}) \xrightarrow{p^{0,1}} H^{0,1}(C) = H^1(C, \mathcal{O}) \ ,$$

and the Real structure $\hat{\iota}$ corresponds to the Real structure defined by the involution $-\iota^* : H^1(C, \mathbb{Z}) \to H^1(C, \mathbb{Z})$ on this quotient. We can now conclude with the following theorem, which describes the image of $\hat{\iota}$-Real line bundle $(\mathcal{L}_\kappa, \tilde{\iota})$ on $\mathrm{Pic}^0(C)$ as element in the fibre product

$$\mathrm{Alt}^2(H^1(C, \mathbb{Z}), \mathbb{Z})^{(\iota^*)^*} \times_{\mathrm{Hom}((\mathrm{id} - \iota^*)H^1(C, \mathbb{Z}), \mathbb{Z}_2)} \mathrm{Hom}(H^1(C, \mathbb{Z})^{-\iota^*}, \mathbb{Z}_2)$$

appearing in our classification Theorem 3.11.

Theorem 4.12. *The element*

$$cw_0([\mathcal{L}_\kappa, \tilde{\iota}]) \in \mathrm{Alt}^2(H^1(C, \mathbb{Z}), \mathbb{Z})^{(\iota^*)^*} \times_{\mathrm{Hom}((\mathrm{id} - \iota^*)H^1(C, \mathbb{Z}), \mathbb{Z}_2)} \mathrm{Hom}(H^1(C, \mathbb{Z})^{-\iota^*}, \mathbb{Z}_2)$$

is the pair (u_C, w_κ), where $w_\kappa : H^1(C, \mathbb{Z})^{-\iota^*} \to \mathbb{Z}_2$ is defined as the unique extension of $f_{u_C} : (\mathrm{id} - \iota^*)H^1(C, \mathbb{Z}) \to \mathbb{Z}_2$ which satisfies the equalities

$$(15) \qquad\qquad w_\kappa([C_i]^\vee) = \langle w(\kappa), [C_i] \rangle + 1 \pmod 2 \ .$$

Proof. Recall that $w_\kappa : H^1(C, \mathbb{Z})^{-\iota^*} \to \mathbb{Z}_2$ is given by the Stiefel-Whitney class of the restriction $\mathcal{L}_\kappa^{\tilde{\iota}}|_{T_0}$ of the real line bundle $\mathcal{L}_\kappa^{\tilde{\iota}}$ to the standard connected component T_0 of $\mathrm{Pic}^0(C)^{\hat{\iota}}$. $\mathcal{L}_\kappa$ possesses a Hermitian-Einstein metric h, which is $\tilde{\iota}$-anti-unitary. Therefore the Hermitian-Einstein connection A_κ on $\mathcal{L}_\kappa$ is compatible with $\tilde{\iota}$, and hence the Stiefel-Whitney class of $\mathcal{L}_\kappa^{\tilde{\iota}}$ is given by the holonomy of this connection along loops contained in $\mathrm{Pic}^0(C)^{\hat{\iota}}$ (see Remark 3.7). On the other hand, by Theorem 4.1 we can read the factor of automorphy (and hence the holonomy along standard loops) of a Yang-Mills connection gauge equivalent to A_κ from the canonical factor of automorphy of $\mathcal{L}_\kappa$.

We apply now Proposition 4.4 which computes this factor of automorphy in terms of Mumford's theta form q_κ and Theorem 4.5 which gives a geometric interpretation for $q_\kappa([C_i]^\vee)$. This proves that $w_\kappa([C_i]^\vee) = \langle w(\kappa), [C_i] \rangle + 1$, as claimed. On the other hand we know, by the results in section 3.3, that w_κ extends f_{u_C}. Finally, by Lemma 4.11 the classes $[C_i]^\vee$ generate $H^1(C, \mathbb{Z})^{-\iota^*}$ modulo the subgroup $(\mathrm{id} - \iota^*)H^1(C, \mathbb{Z})$ of trivial anti-invariants, so (15) determines the extension w_κ. ∎

Remark 4.13.   (1) *The intersection $\langle [C_1]^\vee, \dots, [C_r]^\vee \rangle \cap (\mathrm{id} - \iota^*)H^1(C, \mathbb{Z})$ is not trivial (see Lemma 4.11). The map f_{u_C} agrees with the map defined by the right hand side of (15) on the intersection. This follows from our results, but can also be checked directly.*

    (2) *Using Theorem 4.12 and the difference formula given by Proposition 3.8 we get the Stiefel-Whitney classes of the restrictions of the real line bundle $\mathcal{L}_\kappa^{\tilde{\iota}}$ to all connected components of $\mathrm{Pic}^0(C)^{\hat{\iota}}$.*

5. Real determinant line bundles

Let (C, ι) be a Klein surface with $C^\iota \neq \emptyset$. We have seen that ι induces a Real structure (anti-holomorphic involution) $\hat{\iota} : \mathrm{Pic}(C) \to \mathrm{Pic}(C)$ on the Picard group of C by

$$\hat{\iota}([\mathcal{L}]) = [\iota^*(\bar{\mathcal{L}})] .$$

This involution leaves the degree invariant, so it induces an anti-holomorphic involution on any connected component $\mathrm{Pic}^d(C)$. The geometric theta divisor $\Theta \subset \mathrm{Pic}^{g-1}(C)$ defines a $\hat{\iota}$-invariant holomorphic line bundle $[\mathcal{O}_{\mathrm{Pic}^{g-1}(C)}(\Theta)]$.

For every degree $d \in \mathbb{Z}$ we denote by $\hat{\hat{\iota}}$ the anti-holomorphic involution induced by $\hat{\iota}$ on $\mathrm{Pic}(\mathrm{Pic}^d(C))$. Note that $\hat{\hat{\iota}}$ maps $\mathrm{Pic}^c(\mathrm{Pic}^d(C))$ onto $\mathrm{Pic}^{-\hat{\iota}^*(c)}(\mathrm{Pic}^d(C))$ for every Chern class $c \in \mathrm{NS}(\mathrm{Pic}^d(C))$.

For every $\lambda \in \mathrm{Pic}^{g-1}(C)$ we denote by $\Theta - \lambda \subset \mathrm{Pic}^0(C)$ the $-\lambda$-translate of the geometric theta divisor Θ, and we denote by

$$\mathcal{L}_\lambda := \mathcal{O}_{\mathrm{Pic}^0(C)}(\Theta - \lambda) = (\otimes \lambda)^* \mathcal{O}_{\mathrm{Pic}^g(C)}(\Theta)$$

the corresponding line bundle on $\mathrm{Pic}^0(C)$. The Chern class of $\mathcal{L}_\lambda$ is the element of $H^2(\mathrm{Pic}^0(C), \mathbb{Z}) = \mathrm{Alt}^2(H^1(C, \mathbb{Z}), \mathbb{Z})$ defined by the cup form u_C of C. The assignment $\lambda \mapsto [\mathcal{L}_\lambda]$ defines a holomorphic map

$$\varphi : \mathrm{Pic}^{g-1}(C) \to \mathrm{Pic}^{u_C}(\mathrm{Pic}^0(C)) .$$

Note that $-\hat{\iota}^*(u_C) = u_C$, so the involution $\hat{\hat{\iota}}$ leaves $\mathrm{Pic}^{u_C}(\mathrm{Pic}^0(C))$ invariant.

Lemma 5.1. *The map $\varphi : (\mathrm{Pic}^{g-1}(C), \hat{\iota}) \to (\mathrm{Pic}^{u_C}(\mathrm{Pic}^0(C)), \hat{\hat{\iota}})$ is an isomorphism of Real complex manifolds.*

Proof. For an element $\lambda_0 \in \mathrm{Pic}^0(C)$ one has

$$\varphi(\lambda_0 \otimes \lambda) = (\otimes(\lambda_0 \otimes \lambda))^* \mathcal{O}_{\mathrm{Pic}^g(C)}(\Theta) = (\otimes\lambda_0)^*(\otimes\lambda)^* \mathcal{O}_{\mathrm{Pic}^g(C)}(\Theta) = (\otimes\lambda_0)^* \varphi(\lambda) ,$$

which shows that φ commutes with the natural $\mathrm{Pic}^0(C)$-actions on $\mathrm{Pic}^{g-1}(C)$ and $\mathrm{Pic}^{u_C}(\mathrm{Pic}^0(C))$. The first manifold is obviously a $\mathrm{Pic}^0(C)$-torsor, whereas the second is also a $\mathrm{Pic}^0(C)$-torsor because u_C is a principal polarization of the torus $\mathrm{Pic}^0(C)$ (see [5], p. 36-37). This proves that φ is an isomorphism. On the other hand note that the Real structure $\hat{\hat{\iota}}$ induced on $\coprod_{d \in \mathbb{Z}} \mathrm{Pic}(\mathrm{Pic}^d(C))$ by $\hat{\iota}$ satisfies the identity

$$\hat{\hat{\iota}}((\otimes\lambda)^*(\mathscr{L})) = (\otimes\hat{\iota}(\lambda))^*(\hat{\hat{\iota}}(\mathscr{L})) .$$

Since Θ is $\hat{\iota}$-invariant, it follows that the holomorphic line bundle $\mathcal{O}_{\mathrm{Pic}^{g-1}(C)}(\Theta)$ is $\hat{\hat{\iota}}$-invariant, so for any $\lambda \in \mathrm{Pic}^{g-1}(C)$ one has

$$\varphi(\hat{\iota}(\lambda)) = (\otimes\hat{\iota}(\lambda))^* \mathcal{O}_{\mathrm{Pic}^g(C)}(\Theta) = (\otimes\hat{\iota}(\lambda))^*(\hat{\hat{\iota}}(\mathcal{O}_{\mathrm{Pic}^g(C)}(\Theta)) = \hat{\hat{\iota}}\left((\otimes\lambda)^* \mathcal{O}_{\mathrm{Pic}^{g-1}(C)}(\Theta)\right)$$

$$= \hat{\hat{\iota}}\left((\otimes\lambda)^* \mathcal{O}_{\mathrm{Pic}^{g-1}(C)}(\Theta)\right) = \hat{\hat{\iota}}(\varphi(\lambda)) ,$$

which proves that φ is Real. ∎

We are interested in the induced bijection

$$\pi_0(\varphi) : \pi_0(\mathrm{Pic}^{g-1}(C)^{\hat{\iota}}) \longrightarrow \pi_0(\mathrm{Pic}^{u_C}(\mathrm{Pic}^0(C))^{\hat{\hat{\iota}}}) .$$

We know by Proposition 2.12 that $\pi_0(\mathrm{Pic}^{g-1}(C)^{\hat{\iota}})$ can be identified via the map F_C with the subset of $H^1_{\mathbb{Z}_2}(C, \underline{S}^1(1))$ consisting of isomorphism classes of ι-Real

Hermitian line bundles $(L, \tilde{\iota})$ with $\deg(L) = g - 1$. Using Theorem 2.8 we see that this subset can be identified with

$$H^1(C^\iota, \mathbb{Z}_2)_{g-1} := \{w \in H^1(C^\iota, \mathbb{Z}_2) | \deg_{\mathbb{Z}_2}(w) = g - 1 \pmod 2\} .$$

The composition of these identifications yields a bijection

$$w_C^{g-1} : \pi_0(\mathrm{Pic}^{g-1}(C)^{\hat{\iota}}) \to H^1(C^\iota, \mathbb{Z}_2)_{g-1}$$

which can be explicitly described as follows: for a $\hat{\iota}$-invariant holomorphic line bundle $\mathcal{L}$ of degree $g - 1$, we consider an anti-holomorphic ι-Real structure $\tilde{\iota}$ on $\mathcal{L}$. Then w_C^{g-1} maps the connected component of $[\mathcal{L}]$ in $\mathrm{Pic}^{g-1}(C)^{\hat{\iota}}$ to $w_1(\mathcal{L}^{\tilde{\iota}}) \in H^1(C^\iota, \mathbb{Z}_2)_{g-1}$.

Similarly, the set $\pi_0(\mathrm{Pic}^{u_C}(\mathrm{Pic}^0(C))^{\hat{\hat{\iota}}})$ can be identified via the map $F_{\mathrm{Pic}^0(C)}$ with the subset of the group $H^1_{\mathbb{Z}_2}(\mathrm{Pic}^0(C), \underline{S}^1(1))$ consisting of $\hat{\iota}$-Real Hermitian line bundles $(\mathcal{M}, \tilde{\hat{\iota}})$ on $\mathrm{Pic}^0(C)$ with $c_1(\mathcal{M}) = q$. Therefore, using the results in section 3.3, we see that the set of $\hat{\iota}$-Real Hermitian line bundles $(\mathcal{M}, \tilde{\hat{\iota}})$ on $\mathrm{Pic}^0(C) = H^1(C, \mathcal{O})/H^1(C, \mathbb{Z})$ with $c_1(\mathcal{M}) = u_C$ can be identified with

$$W(u_C) = \{w \in \mathrm{Hom}(H^1(C, \mathbb{Z})^{-\iota^*}, \mathbb{Z}_2) | \ w|_{(\mathrm{id}-\iota^*)H^1(C,\mathbb{Z})} = f_{u_C}\} .$$

Note that the condition $w|_{(\mathrm{id}-\iota^*)H^1(C,\mathbb{Z})} = f_{u_C}$ simply means

(16) $$w(\lambda - \iota^*(\lambda)) = \langle \lambda, \iota^*(\lambda) \rangle \pmod 2 \ \ \forall \lambda \in H^1(C, \mathbb{Z}) .$$

Composing these identifications we obtain a bijection

$$w_{\mathrm{Pic}^0(C)}^{u_C} : \pi_0(\mathrm{Pic}^{u_C}(\mathrm{Pic}^0(C))^{\hat{\hat{\iota}}}) \to W(u_C)$$

which can be explicitly described as follows: for a $\hat{\hat{\iota}}$-invariant holomorphic line bundle $\mathcal{M}$ on $\mathrm{Pic}^0(C)$ with Chern class u_C consider an anti-holomorphic $\hat{\iota}$-Real structure $\tilde{\hat{\iota}}$ on $\mathcal{M}$. Then $w_{\mathrm{Pic}^0(C)}^{u_C}$ maps the connected component of $[\mathcal{M}]$ in $\mathrm{Pic}^{u_C}(\mathrm{Pic}^0(C))^{\hat{\hat{\iota}}}$ to $w_1(\mathcal{M}^{\tilde{\hat{\iota}}}|_{T_0})$, where T_0 denotes (as in section 3.3) the standard connected component of the fixed point locus $\mathrm{Pic}^0(C)^{\hat{\iota}}$.

Concluding, we obtain a diagram

(17)
$$
\begin{array}{ccc}
\pi_0(\mathrm{Pic}^{g-1}(C)^{\hat{\iota}}) & \xrightarrow{\simeq \pi_0(\varphi)} & \pi_0(\mathrm{Pic}^{u_C}(\mathrm{Pic}^0(C))^{\hat{\hat{\iota}}}) \\
\simeq \downarrow w_C^{g-1} & & \simeq \downarrow w_{\mathrm{Pic}^0(C)}^{u_C} \\
H^1(C^\iota, \mathbb{Z}_2)_{g-1} & \xdashrightarrow{\Phi} & W(u_C)
\end{array}
$$

with a bijective upper horizontal arrow and bijective vertical arrows.

Proposition 5.2. *The induced bijection $\Phi : H^1(C^\iota, \mathbb{Z}_2)_{g-1} \to W(u_C)$ is given by the following rule:*

For every $w \in H^1(C^\iota, \mathbb{Z}_2)_{g-1}$, the element $\Phi(w) \in W(u_C)$ is the unique extension of f_{u_C} satisfying the equalities

$$\Phi(w)([C_i]^\vee) = \langle w, [C_i]_2 \rangle + 1 ,$$

where $C_1, \ldots C_r$ are the connected components of C^ι.

Proof. Let $w \in H^1(C^\iota, \mathbb{Z}_2)_{g-1}$ and let $\Gamma := (w_C^{g-1})^{-1}(w) \in \pi_0(\mathrm{Pic}^{g-1}(C)^{\hat{\iota}})$ be the corresponding connected component of $\mathrm{Pic}^{g-1}(C)^{\hat{\iota}}$. We know that any connected component of $\mathrm{Pic}^{g-1}(C)^{\hat{\iota}}$ contains 2^g Real theta characteristics (see [9] p. 169), in

particular we can find a Real theta-characteristic $\kappa \in \Gamma$. Using the notations of section 4.3 we can write $w = w(\kappa)$.

Note now that $\varphi(\kappa) = [\mathcal{L}_\kappa]$, where $\mathcal{L}_\kappa := \mathcal{O}_{\mathrm{Pic}^0(C)}(\Theta_\kappa)$ is the holomorphic line bundle associated with the divisor Θ_κ (see section 4.3).

Therefore $\pi_0(\varphi)((w_C^{g-1})^{-1}(w))$ is the connected component of $[\mathcal{L}_\kappa]$ in the fixed point locus $\mathrm{Pic}^{u_C}(\mathrm{Pic}^0(C))^{\hat{\imath}}$, and $\Phi(w)$ is the element of $W(u_C)$ defined by the Stiefel-Whitney class of $\tilde{\mathcal{L}}_\kappa^{\hat{\imath}}|_{T_0}$, where $\tilde{\imath}$ is the standard $\hat{\imath}$-Real structure of $\mathcal{L}_\kappa$ (see section 4.3), and where T_0 is the standard connected component of the fixed point locus $\mathrm{Pic}^0(C)^{\hat{\imath}}$. It suffices to apply Theorem 4.12. ∎

Corollary 5.3. *Let $p_0 \in C^\iota$. Then $\xi := [\mathcal{O}_C((g-1)p_0)] \in \mathrm{Pic}^{g-1}(C)^{\hat{\imath}}$, and*

$$cw_0([\mathcal{L}_\xi, \tilde{\imath}]) \in \mathrm{Alt}^2(H^1(C, \mathbb{Z}), \mathbb{Z})^{(\iota^*)^*} \times_{\mathrm{Hom}((\mathrm{id}-\iota^*)H^1(C,\mathbb{Z}),\mathbb{Z}_2)} \mathrm{Hom}(H^1(C, \mathbb{Z})^{-\iota^*}, \mathbb{Z}_2)$$

is (u_C, w_{p_0}), where $w_{p_0} \in W(u_C)$ is the unique extension of f_{u_C} satisfying the equalities:

$$w_{p_0}([C_i]^\vee) = \begin{cases} 1 & \text{if} \quad p_0 \notin C_i \\ g \ (mod \ 2) & \text{if} \quad p_0 \in C_i \end{cases}$$

Note that for $\xi := [\mathcal{O}_C((g-1)p_0)] \in \mathrm{Pic}^{g-1}(C)^{\hat{\imath}}$ the Real line bundle $(\mathcal{L}_\xi, \tilde{\imath})$ is just the Real line bundle $\mathcal{O}_{\mathrm{Pic}^0(C)}(\Theta - [\mathcal{O}_C((g-1)p_0)])$ considered in section 1. According to Proposition 1.7 the Stiefel-Whitney of the associated real line bundle on $\mathrm{Pic}^0(C)^{\hat{\imath}}$ controls the orientability of the components of $S^d(C)^\iota$ (for $d > 2(g-1)$). Therefore Corollary 5.3 together with the difference formula given by Proposition 3.8 solves completely the orientability problem formulated in section 1.

We conclude with our final result which solves completely the problems formulated in the introduction and in section 1 about Real determinant line bundles of families of Dolbeault operators:

Theorem 5.4. *Let (C, ι) be a Klein surface with $C^\iota \neq \emptyset$, L a differentiable line bundle of degree d on C, and $\kappa \in \mathrm{Pic}^{g-1}(C)^{\hat{\imath}}$ a Real theta characteristic. Fix a point $p_0 \in C^\iota$, denote by $\delta_{p_0}^L$, $\bar{\partial}_{\kappa,p_0}$ the corresponding families of Dolbeault operators parameterized by $\mathrm{Pic}^d(C)$ and $\mathrm{Pic}^0(C)$ respectively, and by $\det \mathrm{ind}\, \bar{\partial}_{\kappa,p_0}$, $\det \mathrm{ind}\, \delta_{p_0}^L$ the corresponding determinant line bundles endowed with the $\hat{\imath}$-Rea structures given by Remarks 1.4, 1.6.*

(1) The element

$$cw_0([\det \mathrm{ind}\, \bar{\partial}_{\kappa,p_0}]) \in$$

$$\mathrm{Alt}^2(H^1(C, \mathbb{Z}), \mathbb{Z})^{(\iota^*)^*} \times_{\mathrm{Hom}((\mathrm{id}-\iota^*)H^1(C,\mathbb{Z}),\mathbb{Z}_2)} \mathrm{Hom}(H^1(C, \mathbb{Z})^{-\iota^*}, \mathbb{Z}_2)$$

is (u_C, w_κ), where $w_\kappa : H^1(C, \mathbb{Z})^{-\iota^*} \to \mathbb{Z}_2$ is the element of $W(u_C)$ defined as the unique extension of $f_{u_C} : (\mathrm{id} - \iota^*)H^1(C, \mathbb{Z}) \to \mathbb{Z}_2$ which satisfies the equalities

$$w_\kappa([C_i]^\vee) = \langle w(\kappa), [C_i]\rangle + 1 \quad (mod \ 2) \ .$$

(2) Let $\otimes \mathcal{O}_C(dp_0) : (\mathrm{Pic}^0(C), \hat{\imath}) \to (\mathrm{Pic}^d(C), \hat{\imath})$ be the isomorphism of Real complex manifolds defined by $\otimes \mathcal{O}_C(dp_0)$. Then the element

$$cw_0([\{\otimes \mathcal{O}_C(dp_0)\}^* (\det \mathrm{ind}\, \delta_{p_0}^L)]) \in$$

$$\mathrm{Alt}^2(H^1(C, \mathbb{Z}), \mathbb{Z})^{(\iota^*)^*} \times_{\mathrm{Hom}((\mathrm{id}-\iota^*)H^1(C,\mathbb{Z}),\mathbb{Z}_2)} \mathrm{Hom}(H^1(C, \mathbb{Z})^{-\iota^*}, \mathbb{Z}_2)$$

is (u_C, w_{p_0}), where $w_{p_0} \in W(u_C)$ is the unique extension of f_{u_C} satisfying
the equalities:

$$w_{p_0}([C_i]^\vee) = \left\{ \begin{array}{cc} 1 & \text{if} \quad p_0 \notin C_i \\ g \ (mod \ 2) & \text{if} \quad p_0 \in C_i \end{array} \right.$$

Proof. 1. This follows Remark 1.5 and Theorem 4.12.

2. This follows Remark 1.6 and Corollary 5.3.   ∎

6. Appendices

Appendix A: The holonomy formula

Let Y be a differentiable manifold, L a Hermitian line bundle on Y endowed with
a Hermitian connection A and $c : [0, 1] \to Y$ a loop in Y with $c(0) = c(1) = y_0$.
Suppose that c is homotopically trivial, i.e. there exists a smooth map $C : Q \to Y$,
where $Q := [0, 1] \times [0, 1]$, such that

$$C|_R \equiv y_0 \ , \ f(\cdot, 1) = c \ .$$

Here $R := (\{0, 1\} \times I) \cup (I \times \{0\}) = \partial Q \setminus [I \times \{1\}]$. Then

Proposition 6.1. *The holonomy with respect to the connection A along a loop c
can be computed using the formula*

$$h_c^A(\zeta) = e^{\int_Q C^*(F_A)} \zeta \ , \ \forall \zeta \in L_{y_0} \ .$$

Proof. We consider the pull-back connection $B := C^*(A)$ on the pull-back line
bundle $M := C^*(L)$ on the square Q. Since C is constant equal to y_0 on R, one
has an obvious identification

$$M|_R = R \times L_{y_0} \ .$$

Fix $\zeta_0 \in L_{y_0}$ with $\|\zeta_0\| = 1$, and let $\tilde{\zeta}_0 \in \Gamma(R, M)$ be the corresponding constant
section. We extend this section to a section $s_0 \in \Gamma(Q, M)$ with $\|s_0\| \equiv 1$. Let β be
the connection form of B with respect to the trivialization defined by s_0, i.e.

$$\nabla^B s_0 = \beta s_0 \ .$$

Note that the connection B is trivial on R (with respect to the trivialization defined
by $\tilde{\zeta}_0$), so $i_R^*(\beta) = 0$ i.e.

$$(18) \qquad \beta(\frac{\partial}{\partial \theta}) = 0 \text{ on } \{0, 1\} \times I \ , \ \beta(\frac{\partial}{\partial t}) = 0 \text{ on } I \times \{0\} \ .$$

We define now a section $\zeta \in \Gamma(Q, M)$ such that

$$\zeta|_{\{0\} \times I} = \tilde{\zeta}_0|_{\{0\} \times I} \ , \ \nabla^B_{\frac{\partial}{\partial t}} \zeta = 0 \ .$$

In other words, ζ is obtained from $\tilde{\zeta}_0|_{\{0\} \times I}$ by *horizontal parallel transport.* Note
that the map $C \circ \zeta(\cdot, \theta)$ coincides with the parallel transport of $\zeta_0 \in L_{y_0}$ along the
path $c_\theta := C(\cdot, \theta)$ in Y with respect to the connection A, so – taking $\theta = 1$ – we
see that the parallel transport along the given path c corresponds via C to $\zeta(\cdot, 1)$.

Put $\zeta = g s_0$, where $g : Q \to S^1$ is a smooth map. The condition $\nabla^B_{\frac{\partial}{\partial t}} \zeta = 0$
becomes

$$g^{-1} \frac{\partial g}{\partial t} = -\beta(\frac{\partial}{\partial t}) \ .$$

Consider the 1-form η on Q defined by

$$\eta := g^{-1}\frac{\partial g}{\partial \theta}d\theta = \frac{\partial}{\partial \theta}(\log g)d\theta \ .$$

One has

$$d\eta = \frac{\partial^2}{\partial t \partial \theta}(\log g)dt \wedge d\theta = -\frac{\partial}{\partial \theta}\beta(\frac{\partial}{\partial t})dt \wedge d\theta = d\beta - \frac{\partial}{\partial t}\beta(\frac{\partial}{\partial \theta})dt \wedge d\theta =$$

$$= d\beta - d(\beta(\frac{\partial}{\partial \theta})d\theta) \ .$$

Therefore

$$d(\eta + \beta(\frac{\partial}{\partial \theta})d\theta) = d\beta = F_B = C^*(F_A) \ .$$

Using Stokes formula, we obtain

$$\int_{\partial Q} \eta + \beta(\frac{\partial}{\partial \theta})d\theta = \int_Q C^*(F_A) \ .$$

But one has $\beta(\frac{\partial}{\partial \theta})d\theta)\big|_{\partial Q} = 0$ by (18) and $\eta|_{I \times \{0,1\} \cup \{0\} \times I} = 0$ by the definition of η. So the left hand term reduces to the integral of η on the segment $\{1\} \times I$. This gives

$$\int_0^1 \frac{\partial}{\partial \theta} \log(g(1,\theta))d\theta = \int_Q C^*(F_A) \ ,$$

so

$$g(1,1) = g(1,1)g(1,0)^{-1} = \int_Q C^*F_A \ .$$

But $h_c(\zeta_0) = g(1,1)\zeta_0$, which proves the formula. ∎

Corollary 6.2. *Let c_0, $c_1 : I \to Y$ be to paths in Y with $c_i(0) = y_0$, $c_i(1) = y_1$. Let $C : Q \to Y$ be a smooth homotopy from c_0 to c_1 in the class of paths with fixed end points y_0, y_1. The parallel transports with respect to A along c_i compare by a formula of the form*

$$h_{c_1}^A(\zeta) = h(c_0, c_1)h_{c_0}^A(\zeta) \ \forall \zeta \in L_{y_0} \ ,$$

where $h(c_0, c_1) \in S^1$. Then it holds

$$h(c_0, c_1) = e^{\int_Q C^*(F_A)} \ .$$

Appendix B: Identities for the Stiefel-Whitney numbers of
Real vector bundles

Let (X, ι) be a closed, connected, oriented differentiable n-manifold endowed with an involution ι, X^ι the fixed point locus of ι, and let $(E, \tilde{\iota})$ be a ι-Real vector bundle of rank r on X. We denote by $_\mathbb{R}E$ the underlying real bundle of E, and by $E^{\tilde{\iota}}$ the fixed point locus of $\tilde{\iota}$, regarded as a real bundle of rang r on X^ι.

Remark 6.3. *Let $X \supset Y \xrightarrow{\theta} X^\iota$ be the projection (with $\mathbb{Z}_2$-invariant fibres) of a sufficiently small tubular neighborhood Y of X^ι. Then the restriction $(E, \tilde{\iota})\big|_Y$ can be identified with $\theta^*(E^{\tilde{\iota}}) \otimes \mathbb{C}$ with the involution defined by conjugation.*

Proof. The map of $\mathbb{Z}_2$-spaces $\theta : Y \to X^\iota$ is a $\mathbb{Z}_2$-homotopy equivalence, so the result follows from [2]. ∎

Suppose n is even, and let $\mathfrak{c} = \Pi_{i=1}^{r} c_i^{k_i}$ be a Chern monomial of degree $n = 2\sum ik_i = n$, and let $\mathfrak{w} = \Pi_{i=1}^{r} w_{2i}^{k_i}$ be the associated Stiefel-Whitney monomial. We will show that the corresponding Stiefel-Whitney number

$$\langle \mathfrak{w}(_{\mathbb{R}}E), [X]_2 \rangle \equiv \langle \mathfrak{c}(E), [X] \rangle \pmod 2$$

can be computed by a polynomial expression in the Stiefel-Whitney classes of the real bundle $E^{\tilde{\iota}}$ over X^{ι} and the normal bundle of X^{ι} in X.

Consider the $(n+1)$-dimensional fibre product $Q := X \times_{\mathbb{Z}_2} S^1$, where $\mathbb{Z}_2$ acts on X via τ and on S^1 via the antipodal involution. This fibre product can be regarded as the total space of the locally trivial bundle $p : Q \to \mathbb{P}^1_{\mathbb{R}}$ with fibre X associated with the principal $\mathbb{Z}_2$-bundle $S^1 \to \mathbb{P}^1_{\mathbb{R}}$. By definition Q can be also regarded as the base of a principal $\mathbb{Z}_2$-bundle $q : X \times S^1 \to X$. We define a $\mathbb{Z}_2 = \{\pm 1\}$-action on Q by

$$(-1) \cdot [x, x_0, x_1]) := [x, x_0, -x_1] = [-x, -x_0, x_1] .$$

This action lifts the $\mathbb{Z}_2$-action on $\mathbb{P}^1_{\mathbb{R}}$ induced by the standard orientation reversing reflection with fixed points $0 = [1, 0]$, $\infty = [0, 1] \in \mathbb{P}^1_{\mathbb{R}}$. Q decomposes as a union of two $\mathbb{Z}_2$-invariant open set $Q_0 \supset p^{-1}(0)$, $Q_\infty \supset p^{-1}(\infty)$ both diffeomorphic to $X \times \mathbb{R}$ via the maps

$$(x, x_1) \overset{j_0}{\mapsto} [x, 1, x_1] , \;\; (x, x_0) \overset{j_\infty}{\mapsto} [x, x_0, 1] .$$

The induced actions on $X \times \mathbb{R}$ are

$$(x, x_1) \mapsto (x, -x_1) , \;\; (x, x_0) \mapsto (-x, -x_0) .$$

The fixed point locus $Q^{\mathbb{Z}_2}$ of the $\mathbb{Z}_2$-action on Q decomposes as

$$Q^{\mathbb{Z}_2} = Q_0^{\mathbb{Z}_2} \cup Q_\infty^{\mathbb{Z}_2} ,$$

where $Q_0^{\mathbb{Z}_2} = p^{-1}(0)$ is naturally isomorphic with X, and $Q_\infty^{\mathbb{Z}_2} \subset p^{-1}(\infty)$ can be identified with X^{ι}. The normal bundle of $Q_0^{\mathbb{Z}_2} \simeq X$ in Q is $X \times \mathbb{R}(1)$ (trivial line bundle with the $\mathbb{Z}_2$-action induced by $-\mathrm{id}$), whereas the normal bundle of $Q_\infty^{\mathbb{Z}_2} \simeq X^{\iota}$ in Q is $N_{X^{\iota}/X}(1) \times \mathbb{R}(1)$.

Let $U := U_0 \coprod U_\infty$ be an open $\mathbb{Z}_2$-equivariant tubular neighborhood of $Q^{\mathbb{Z}_2}$. The quotient

$$W := {}^{[Q \setminus U]}\big/_{\mathbb{Z}_2}$$

is a compact manifold with boundary

$$\partial W = X \coprod \mathbb{P}_{\mathbb{R}}(N_{X^{\iota}/X} \oplus \mathbb{R})$$

consisting of X and of the real projectivization of the normal bundle of X^{ι} in Q. The bundle $p_1^*(_{\mathbb{R}}E)$ on $X \times S^1$ comes with an obvious $\mathbb{Z}_2$-action and descends to Q via q, because q is the quotient with respect to a free $\mathbb{Z}_2$-action. The total space of the descended bundle is the fibre product $F = E \times_{\mathbb{Z}_2} S^1$, where $\mathbb{Z}_2$ acts on E via $\tilde{\iota}$. The pull-backs $j_0^*(F)$, $j_\infty^*(F)$ can both be identified with $p_X^*(E) = E \times \mathbb{R}$. The $\mathbb{Z}_2$-action on Q can be lifted to F using the formula

$$(-1) \cdot [e, x_0, x_1]) := [e, x_0, -x_1] = [-e, -x_0, x_1] ,$$

and the induced actions on $j_0^*(F) = E \times \mathbb{R}$, $j_\infty^*(F) = E \times \mathbb{R}$ are

$$(e, x_1) \mapsto (e, -x_1) , \;\; (e, x_0) \mapsto (-e, -x_0) .$$

Using this lift we see that the restriction $F|_{Q \setminus U}$ descends to W. Using the identifications $Q_0^{\mathbb{Z}_2} = X$, $Q_\infty^{\mathbb{Z}_2} = \mathbb{P}_{\mathbb{R}}(N_{X^{\iota}/X} \oplus \mathbb{R})$, denoting by χ the tautological real

line bundle on this projective bundle, and by π the projection $\mathbb{P}_\mathbb{R}(N_{X^\iota/X}\oplus\underline{\mathbb{R}}) \to X^\iota$, we see that the restrictions of the obtained bundle $\bar{F}$ to the two parts of ∂W are

$$\bar{F}\big|_{Q_0^{\mathbb{Z}_2}} = E \ , \ \bar{F}\big|_{Q_\infty^{\mathbb{Z}_2}} = \pi^*(E^{\tilde\iota}) \oplus [\pi^*(E^{\tilde\iota}) \otimes \chi] \ .$$

For the second formula we used Remark 6.3 and the obvious $\mathbb{R}$-isomorphism of $\mathbb{Z}_2$-bundles $E^{\tilde\iota} \otimes \mathbb{C} \simeq E^{\tilde\iota} \oplus E^{\tilde\iota}(1)$.

Applying the Whitney formula one can decompose:

$$\mathfrak{w}\left(\pi^*(E^{\tilde\iota}) \oplus [\pi^*(E^{\tilde\iota}) \otimes \chi]\right) = a_0 w_1(\chi)^n + \sum_{0 < ij \le n} a_{ij} w_i^j(\pi^*(E^{\tilde\iota})) w_1(\chi)^{n-ij} \ .$$

Suppose that X^ι has constant codimension k at any point, and denote by η the real $(k+1)$-bundle $N_{X^\iota/X} \oplus \underline{\mathbb{R}}$ on X^ι. Note that $w_i(\eta) = w_i(N_{X^\iota/X})$, in particular $w_{k+1}(\eta) = 0$. We have

$$\pi_*(w_1(\chi)^{k+l}) = s_l(\eta),$$

where $s(\eta) = \sum_{l=0}^\infty s_l(\eta) = w(\eta)^{-1}$. Hence we obtain

$$\pi_*\left\{\mathfrak{w}\left(\pi^*(E^{\tilde\iota}) \oplus [\pi^*(E^{\tilde\iota}) \otimes \chi]\right)\right\} = a_0 s_{n-k}(\eta) + \sum_{l=1}^{n-k} \sum_{ij=l} a_{ij} w_i^j(E^{\tilde\iota}) s_{n-k-l}(\eta) \ .$$

Regarding W as an homology equivalence between its boundary parts X and $\mathbb{P}_\mathbb{R}(N_{X^\iota/X} \oplus \underline{\mathbb{R}})$, and using the identity

$$\langle \sigma, [\mathbb{P}_\mathbb{R}(N_{X^\iota/X} \oplus \underline{\mathbb{R}})]_2 \rangle = \langle \pi_*(\sigma), [X^\iota]_2 \rangle$$

we get the following *localization formula*:

Theorem 6.4. *Let (X, ι) be a closed, connected, oriented differentiable n-manifold endowed with an involution ι, X^ι the fixed point locus of ι, and let $(E, \tilde\iota)$ be a ι-Real vector bundle of rank r on X. Then for every Chern monomial $\mathfrak{c}$ of degree n and corresponding Stiefel-Whitney monomial $\mathfrak{w}$ we have*

$$\langle \mathfrak{w}(_\mathbb{R}E), [X]_2 \rangle = \left\langle a_0 s_{n-k}(\eta) + \sum_{l=1}^{n-k} \sum_{ij=l} a_{ij} w_i^j(E^{\tilde\iota}) s_{n-k-l}(\eta), [X^\iota]_2 \right\rangle \ .$$

Example: $n = 2$, $k = 1$, $\mathfrak{w} = w_2$. In this case one has:

$$w_2\left(\pi^*(E^{\tilde\iota}) \oplus [\pi^*(E^{\tilde\iota}) \otimes \chi]\right) = w_1(\pi^*(E^{\tilde\iota}))(w_1(\pi^*(E^{\tilde\iota})) + r w_1(\chi)) + w_2(\pi^*(E^{\tilde\iota}) \otimes \chi) +$$

$$+ w_2(\pi^*(E^{\tilde\iota})) = w_1(\pi^*(E^{\tilde\iota})) w_1(\chi) + \frac{r(r-1)}{2} w_1^2(\chi) + w_1(\pi^*(E^{\tilde\iota}))^2 =$$

$$= w_1(\pi^*(E^{\tilde\iota})) w_1(\chi) + \frac{r(r-1)}{2} w_1^2(\chi) \ .$$

Here we used have the general formula:

$$w_2(F \otimes \chi) = w_2(F) + (r-1) w_1(F) w_1(\chi) + \frac{r(r-1)}{2} w_1^2(\chi) \ .$$

Therefore

Corollary 6.5. *Let $(E, \tilde{\iota})$ be a ι-Real vector bundle of rank r on a closed Real 2-manifold (X, ι) with X^ι of codimension 1. Then*

$$\langle w_2(\mathbb{R}E), [X]_2 \rangle = \langle w_1(E^{\tilde{\iota}}) + \frac{r(r-1)}{2} w_1(N_{X^\iota/X}), [X^\iota]_2 \rangle .$$

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